Eo2a44

CHAMBERS'S EDUCATIONAL COURSE—EDITED BY W. AND R. CHAMBERS.

A KEY

TO THE

ELEMENTS OF PLANE GEOMETRY

IN

"CHAMBERS'S EDUCATIONAL COURSE;"

CONTAINING

SOLUTIONS OF ALL THE EXERCISES IN THAT WORK.

WILLIAM AND ROBERT CHAMBERS,
LONDON AND EDINBURGH.
1861.

Edinburgh:
Printed by W. and R. Chambers.

PREFACE.

THE following pages contain Solutions of the Exercises given in the "Elements of Plane Geometry" in Chambers's Educational Course—forming what is usually termed a KEY to that volume. Such a Key will be of utility to those privately prosecuting the study of Geometry; and when consulted only after the Student has made every effort from his own knowledge to solve an Exercise, it will be found to supply in some measure the office of an Instructor. It will likewise be of advantage to those Teachers whose school-room duties are so varied and miscellaneous as to prevent them from bestowing that time and study which Mathematical Solutions in general require. Nor will the volume be without its use to every class of Geometrical Scholars, as it may be read as a sequel to the ordinary course of study contained in the Elements of Euclid.

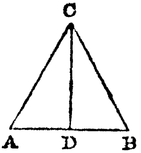
Several of the Exercises given are original; and in the Solutions of all, the Author has endeavoured to be as plain and explicit as possible—adopting methods more simple and concise than are to be found in most collections of a similar nature.

KEY TO PLANE GEOMETRY.

FIRST BOOK.

EXERCISE I.—THEOREM.

A LINE that bisects the vertical angle of an isosceles triangle, also bisects the base perpendicularly.

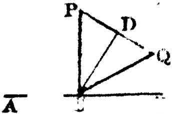


Let ABC be an isosceles triangle, of which the vertical angle ACB is bisected by the line CD, then AB is bisected in D, and CD is perpendicular to AB.

For AC = CB (I. Def. 26), and CD is common to the two triangles ACD, BCD; therefore the two sides AC, CD, are equal to BC, CD, respectively, and the contained angles ACD, BCD, are equal (by hypothesis); hence the triangles are everyway equal (I. 4), and consequently AD = DB, and angle ADC = BDC; but these are adjacent angles; hence they are right, and CD is perpendicular to AB (I. [illegible]

EXERCISE II.—PROBLEM.

In a given straight line, to find a point equally distant from two given points.



Let AB be the given line, and PQ, the given points, to find a point, as C in AB, that shall be equidistant from the points P and Q.

Join P and Q, bisect PQ in D, and draw CD perpendicular to PQ, and produce it to cut AB in C; and then C is the required point.

For in the two triangles CDP, CDQ, PD = DQ, and CD is common, and the angles CDP, CDQ, are equal, being right; consequently (I. 4) the triangles are everyway equal, and CP = CQ; wherefore C is the required point.

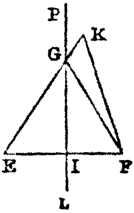
It is evident, when the given points are on opposite sides of the given line, or when one of the points is in the given line, that the same construction and proof will apply as in the above case, when the points are on one side of the line.

When the two points are so situated that the line joining them is perpendicular to the given line, the problem is impossible, unless they happen to be equidistant from the line; for the line bisecting perpendicularly the line joining them, would be parallel to the given line, and would therefore never meet it.

EXERCISE III.—THEOREM.

If a line be bisected perpendicularly by another line, every point in the latter is equally distant from the extremities of the former; and any point not situated in the latter is at unequal distances from the extremities of the former.

Let EF be bisected perpendicularly by PL, then any point G in PL is equidistant from E and F; and a point K, not situated in PL, is at unequal distances from E and F.



For join GE, and GF; then, because EI = IF (by hypothesis), and IG is common to the two triangles EIG, FIG, and the angles at I are equal, being right; therefore (I. 4) the triangles are equal in every respect, and therefore EG = GF; also angle GEI = GFI.

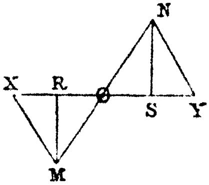
Again, because angle GEI = GFI, therefore KFI, which exceeds GFI, is greater than GEI, and consequently (I. 19) the side KE is greater than KF; or K is unequally distant from E and F.

It may also be proved, by means of I. 20, that KE is greater than KF.

EXERCISE IV.—THEOREM.

If any line be drawn through the middle point of the line joining two given points, any two points in the former line that are equidistant from the middle point, are also equidistant from the two given points.

Let MN be the given points, O the middle of MN, and XY any line through O; also let OX = OY, then will MX = NY.



For in the triangles OMX, ONY, the sides OX, OM, in the one, are respectively equal to OY, ON, in the other, and the vertical angles at O are equal (I. 15); consequently (I. 4) the triangles are everyway equal, and therefore M [illegible]

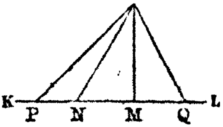
It is evident that, in whatever direction the line XY is drawn, provided it passes through the point O, the same reasoning will apply. This exercise may be considered to be a theoretical porism (see Porism, p. 208, Plane Geometry).

EXERCISE V.—THEOREM.

Of all the lines that can be drawn from a given point to a given line, the perpendicular upon it is the least; and of all others, that which is nearer to the perpendicular is less than one more remote; and only two equal lines can be drawn to it from that point, one upon each side of the perpendicular.

Let KL be a given line, and O a given point, OM a perpendicular on KL, and ON, OP, any other lines; then OM is the least; and ON is [illegible] than OP; also only two equal lines, ON, OQ, can be [illegible] from O to KL, one on each side of the perpendicular OM.

For in triangle OMN, the angle at M is right, and consequently (I. 32, Cor.) the angle at N is less than a right angle; hence (I. 19) ON is greater than OM. In the same manner it is proved that OP or OQ is greater than OM.



Again, angle ONP, being an exterior angle of the triangle OMN, is greater than the interior angles OMN; that is, ONP is obtuse: but in triangle OMP, right-angled at M, the angle OPM must be acute (I. 32, Cor.); consequently angle ONP is greater than OPM, hence (I. 19) OP is greater than ON.

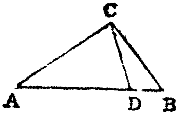
Lastly, make MQ = MN, and join OQ; then, in the triangles OMN, OMQ, the side MN = MQ, and OM is common to them, and the angles at M are equal, being right; hence the triangles are everyway equal, and therefore ON = OQ. And no other line but ON can be drawn equal to OQ; for, if possible, let OP be the line; then, since ON and OP are at unequal distances from the perpendicular OM, they are unequal; but ON = OQ; therefore OP cannot be equal to OQ.

EXERCISE VI.—THEOREM.

The difference between two sides of a triangle is less than the third side.

Let ABC be a triangle; then the difference between any two of its sides, as AB and AC, is less than the third side BC.

For of the two sides AB, AC, let AB be the greater, and from it cut off a part, AD = AC the less, then (I. 20) AC and CB are greater than AB or AD and DB; and if from these unequals the equals AC and AD be taken away, the remainder CB will still be greater than the remainder DB.



When the sides AB, AC, happen to be equal, their difference is nothing, and the proposition is evident; it is also evident when the third side BC is equal to the greater side AB.

EXERCISE VII.—THEOREM.

The perpendiculars drawn from two given points to any line that bisects the line joining the points, are equal.

Let MN be the given points (fig. to Ex. 4), O the middle of MN, XY any line through O, and MR, NS, perpendiculars on XY from M and N, then MR = NS.

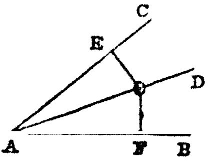
For in the triangles OMR, ONS, the vertical angles at O are equal (I. 15); the right angles at R, S, are equal; and the sides OM, ON, are also equal; hence (I. 26) the two triangles are equal in every respect, and therefore MR =

It is evident that if the line XY passes through the point O in any other direction, that the same proof

EXERCISE VIII.—THEOREM.

Every point in the line that bisects a given angle is equidistant from the sides of the angle.

Let AB, AC, be the sides of an angle BAC, which is bisected by the line AD; then if, from any point O in AD, the perpendiculars OE, OF, be drawn upon the sides, then is OE = OF.



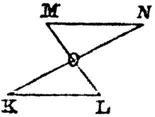
For in the triangles AOE, AOF, the angles at A are equal by construction, the angles at E and F are right, and the side AO is common to the two triangles; consequently (I. 26) they are everyway equal, and therefore OE = OF.

EXERCISE IX.—THEOREM.

If the alternate extremities of two equal and parallel lines be joined, the connecting lines bisect each other.

Let MN, KL, be two equal and parallel lines; then ML *and KN, joining their alternate extremities, bisect each other in O.*

For in the two triangles MON, KOL, the vertical angles at O are equal (I. 15), the alternate angles at M and L are equal, and the sides KL, MN, are also equal; therefore (I. 26) the triangles are equal in every respect, and consequently OK = ON, and OL = OM; that is, ML and KN are bisected in O.

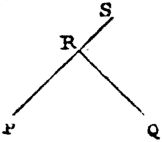


EXERCISE X.—THEOREM.

If the vertical angle of an isosceles triangle be a right angle, each of the angles at the base is half a right angle.

Let PQR be an isosceles triangle, having its vertical angle R right; then each of the angles P and Q, at the base, is half a right angle.

For the three angles P, Q, R, of triangle PQR, are together equal to two right angles, of which R is right; therefore the sum of P and Q is equal to a right angle; and since they are also equal (I. 5), therefore each of them must be half a right angle.



EXERCISE XI.—THEOREM.

If a side of an isosceles triangle be produced beyond the vertex, the exterior angle is double of either of the angles at the base.

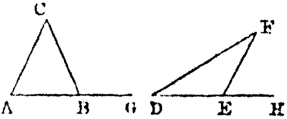
Let PQR (fig. to Ex. 10) be an isosceles triangle, and let the side PR be produced beyond the vertex to S, then the exterior angle QRS is double of either of the angles P or Q.

For (I. 32) the exterior angle QRS is equal to the two interior opposite angles P and Q together; but these angles being equal (I. 5), their sum is double of either of them; consequently the exterior angle QRS is double of angle P or Q.

EXERCISE XII.—THEOREM.

If the exterior angle, and one of the opposite interior angles, in one triangle, be respectively double those of another, the remaining opposite interior angle of the former is double that of the latter.

Let ABC and DEF be two triangles, of which the exterior angle CBG of the former is double the exterior angle FEH of the latter, and the interior angle A of the former double the interior angle D of the latter, then shall angle C of the former be double of F in the latter.



For angle CBG = angles A and C together, also angle FEH = angles D and F together; but angle CBG = twice FEH by hypothesis; consequently angles A and C together must = twice angle D and twice angle F; but angle A is double of D by hypothesis; and taking away these equals from the preceding equal quantities, there remains angle C double of F.

Or more concisely thus:

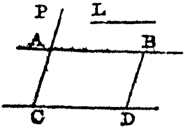
angle CBG = A + C, angle FEH = D + F;
but CBG = twice FEH by hypothesis;
hence A + C = twice D and twice F;
but by hypothesis A = twice D; and taking away these equals from the preceding, there remains angle C = twice angle F.

EXERCISE XIII.—PROBLEM.

Through a given point to draw a line such that the segment of it intercepted between two given parallels may be equal to a given line.

Let P be the point, and AB, CD, the parallels, and L the given line, to draw through P a line PAC, so that AC

Take any point B in one of the parallels, and from it as a centre, with L as a radius, cut CD in D; draw BD, and through P draw PC parallel to BD, and PC is the required line.



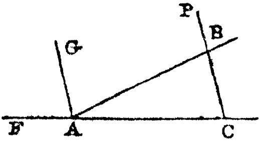
For AB, CD, are given parallel, and PC is parallel to BD; hence the figure AD is a parallelogram, and consequently the side AC is = BD (I. 34).

EXERCISE XIV.—PROBLEM.

Through a given point to draw a line that shall be equally inclined to two given lines.

Let AB, AC, be two given lines, and P a given point, it is required to draw from P a line PC, making equal angles with AB and AC.

Produce CA to F, and bisect the angle BAF by the line AG, and from P draw PC parallel to AG, and it is the required line.



For since AG and PC are parallel, therefore (I. 29) the exterior angle FAG is = ACP, the interior and opposite; and GAB is = ABC, as they are alternate angles; but FAG = GAB by construction, consequently ACP = ABC.

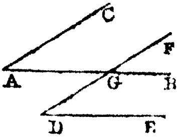
When the point lies between the given lines, or is situated in one of them, it is evident that the same method of solution applies.

EXERCISE XV.—THEOREM.

If two intersecting lines be respectively parallel, or equally inclined, to other two intersecting lines, the inclination of the former is equal to that of the latter.

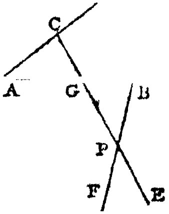
CASE 1.—When two of the intersecting lines are respectively parallel to the other two lines.

Let AC, AB, be two intersecting lines respectively parallel to DF, DE; then angle A is = D.



For produce AB, DF, if necessary, to meet in G; then the alternate angles A and AGD are equal (I. 29), because AC is parallel to DF; and the alternate angles D and AGD are also equal, since AB is parallel to DE; consequently angles A and D being each = AGD, are equal.

CASE 2.—Let the two intersecting lines AC, AG, be respectively inclined to the two PE, PF, at the same angle; then angle A = FPE.



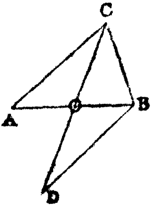
For produce FP, EP, if necessary to meet the other two lines in C and B; then, since the inclinations of FP, EP, to AB, AC, respectively, are equal, therefore angle ACG = GBP; but the vertical angles AGC, PGB, at G, are also equal; consequently the third angles of the triangles AGC, PGB, are equal; that is, angle A = BPG (I. 32, Cor. 3) = FPE (I. 15).

EXERCISE XVI.

The sum of two sides of a triangle is greater than twice the line joining the vertex and the middle of the base.

Let ABC be a triangle, and CO the line joining the vertex and middle of the base; then AC and CB together are greater than twice CO.

For produce CO to D, till OD = OC, and join DB; then the sides AO, OC, are equal to BO, OD, respectively, in the two triangles AOC, BOD, and the contained angles at O are equal; hence (I. 4) the triangles are everyway equal, and therefore AC = DB. But DB and BC together are greater than CD (I. 20); consequently AC and BC are also greater than CD or twice CO.



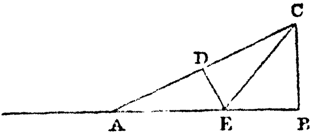
EXERCISE XVII.—PROBLEM.

Given the sum or difference of the hypotenuse and a side of a right-angled triangle, and also the remaining side, to construct it.

Let BC be a side of the right angle of a right-angled triangle, and AB the sum of the hypotenuse and the other side, to construct the triangle.

Let AB, BC, be placed so that BC is perpendicular to AB; join AC, and bisect AC perpendicularly by DE; join EC, and EBC is the required triangle.

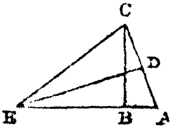
For in the triangles ADE, CDE, the sides AD, DE, are respectively equal to CD, DE, and the angles contained by these sides are right; therefore (I. 4) the triangles are everyway equal, and hence CE = AE; therefore BE and EC together are equal to BA, the given sum, and BC is the given side; therefore CBE is the required triangle.



Again, let BC be the given side, and AB the given difference, to construct the triangle.

Let BC be placed perpendicularly to AB, and join AC; bisect AC perpendicularly by DE, and join CE; then EBC is the required triangle.

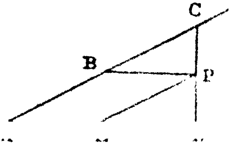
For, as in the preceding figure, the triangles ADE, CDE, are everyway equal, and therefore CE = CA; consequently the difference between CE and EB is equal to that between EA and EB; that is, it is AB. Hence AB is the difference between the hypotenuse EC and the side EB, and BC is the given side; therefore EBC is the required triangle.



EXERCISE XVIII.—PROBLEM.

Through a given point, between two given lines, to draw a line so that the part of it intercepted between them may be bisected in that point.

Let AD, AC, be the given lines, and P the given point.



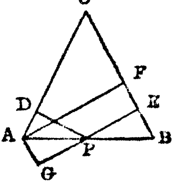
Through P draw PB parallel to AD, make BC = AB; join CP, and produce CP to D, and it is the required line.

For draw PE parallel to AC; then BE is a parallelogram (I. Def. 42), as its opposite sides are parallel; hence PE = AB (I. 34), and AB = BC by construction; therefore PE = BC; also angle EPD = BCP (I. 29) and CBP = BPE, being alternate angles, also BPE = PED for a similar reason; therefore CBP = PED. Consequently the two triangles BCP, EPD, have two angles and a side in the one equal to two angles and a corresponding side in the other; they are therefore everyway equal, and hence CP = PD; and consequently CD is the required line.

EXERCISE XIX.—THEOREM.

If from any point in the base of an isosceles triangle perpendiculars be drawn to the sides, their sum will be equal to the perpendicular from either extremity of the base upon the opposite side.

Let ABC be an isosceles triangle, P any point in its base, PD, PE, perpendiculars on its sides from P, and AF perpendicular to BC; then PD and PE together, are equal to AF.



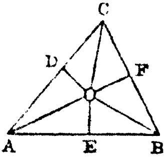
For produce EP, and draw AG perpendicular to it; then AG and FE are parallel, because the interior angles G and FEG are together equal to two right angles (I. 29); for a similar reason AF and GE are parallel; consequently AGEF is a rectangle, and GE = AF. Again, the alternate angles GAP and B are equal, and B = DAP (I. 5); hence GAP = DAP, and the angles at G and D of the triangles APG, APD, are right, and the side AP is common; hence the triangles are equal (I. 26), and therefore PD = PG; and DP, PE, are together equal to GP, PE, together; that is, DP + PE = GE = AF.

EXERCISE XX.—THEOREM.

If two lines bisect the angles at the base of a triangle, the line joining their point of intersection and the vertex bisects the vertical angle.

Let the angles CAB, CBA, of the triangle ABC, be bisected by the lines AO, BO; then OC will bisect the vertical angle ACB.

For draw OD, OE, OF, perpendicular to the three sides; then in the triangles AOD, AOE, the angles at A are equal by construction; the angles at D and E are right, and the side AO is common to both; consequently the triangles are equal (I. 26), and OD = OE. In the same manner it is proved that OF = OE; hence OD = OF. Again, in the two triangles COD, COF, the two sides CO, OD, in one, are respectively equal to CO, OF, in the other; the angles at D and F are right, and hence the angles at C are of the same species (Def. 14 and 32, Cor. 4); consequently (I. C.) the triangles are every-way equal, and therefore angle DCO = FCO; that is, the vertical angle C is bisected.

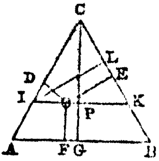


EXERCISE XXI.—THEOREM.

The sum of the perpendiculars drawn from any point within an equilateral triangle on the three sides, is equal to the perpendicular from any of the angular points upon the opposite side.

Let ABC be an equilateral triangle; the perpendiculars OD, OE, OF, upon the sides from any point O within it, are equal to the perpendicular CG.

Through O draw IK parallel to AB, and draw IL perpendicular to BC. Then OF and PG being perpendicular



to AB, are parallel; also IK is parallel to AB; hence OG is a parallelogram, and therefore OF = PG. Again, in the triangles CIL, CIP, angle ICL = CIP, for CIP = A (I. 29); also the right angle ILC = CPI, for CPI = PGF (I. 29); also the side IC is common to the two triangles; they are therefore (I. 26) everyway equal, and consequently IL = CP. But since IK is parallel to AB, angle IKC = B, and B = A (I. 5, Cor.), and it was already shown that CIP = A; hence CKI = CIP, and CIK is an isosceles triangle, and also equilateral; whence (Ex. 19) OD + OE = IL = CP; and adding OF = PG to both these equals, OD + OE + OF = CP + PG = CG.

EXERCISE XXII.—THEOREM.

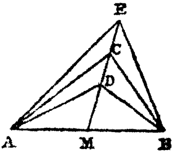
Half the base of a triangle is greater than, equal to, or less than, the line joining the middle of the base and the vertex, according as the vertical angle is obtuse, right, or acute.

1. Let ACB be a right-angled triangle, having ACB for its right angle; then, if M is the middle of the base, MC = AM or BM.

For if not, let MC be greater than AM; then angle CAM is greater than ACM (I. 18), for the same reason MBC is greater than MCB; consequently the two angles CAM and CBM, are together greater than the right angle ACB, and therefore the three angles A, B, C, of the triangle ABC, are together greater than two right angles, which is impossible (I. 32). Therefore MC cannot be greater than AM or MB.

Again, if MC be supposed less than AM or MB, it could be similarly proved that the three angles of the triangle ABC are less than two right angles, which is impossible; therefore MC is not less than AM or MB; and it was proved above that it is not greater; hence CM = AM = BM.

2. Let AEB, the vertical angle of the triangle ABE, be acute, and M the middle of the base; then ME is greater than AM.



For at some point C, in the line ME, below E, the angle AEB, formed by lines drawn to A or B, will be a right angle (I. 21), and by the first case AM = MC; hence AM is less than ME.

3. Let ADB, the vertical angle of the triangle ABD, be obtuse, and MD the line joining the vertex and middle of the base; then AM is greater than MD.

For at some point C, in MD produced, the angle ACB, formed by lines drawn to A and B from C, will be a right angle (I. 21 and 32); and by the first case, AM = MC; consequently AM is greater than DM.

The second and third cases could be proved indirectly like the first. Thus, if in the second, AM be supposed = ME, it would follow that angle AEB would be equal to the two angles EAB, EBA, and therefore that it would be a right angle, being half the sum of the three angles (I. 32), which is contrary to the hypothesis of its being acute. And if AM were supposed to be greater than ME, it would follow that angle AEB would exceed the two angles EAB and EBA, and therefore (I. 32) that it would be obtuse, which is also contrary to hypothesis. Hence if AM is neither = ME, nor greater than ME, it must be less.

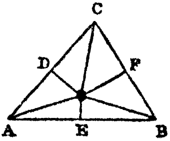
In exactly the same manner it can be proved, when angle ADB is obtuse, that AM is neither = MD, nor less than MD; and consequently it must be greater than MD.

EXERCISE XXIII.—THEOREM.

If two lines bisect perpendicularly two sides of a triangle, the perpendicular from their point of intersection upon the base will bisect it.

Let ABC be a triangle, two of whose sides AC, BC, are bisected perpendicularly by the lines DO, FO, meeting in O; then OE, being drawn perpendicularly to AB, will bisect it in E.

In the two triangles AOD, COD, the sides AD, DO, are equal, and OD is common to the two triangles ADO, CDO; also the contained angles ADO, CDO, are right; consequently (I. 4) the triangles are equal in every respect, and hence $AO = OC$. Comparing the two triangles BOF, COF, it can in the same way be proved that the side $BO = OC$. But it was proved above that $AO = OC$; consequently $AO = OB$. Again, in the triangles AEO, BEO, the side $AO = OB$; therefore angle $OAE = OBE$, and the angles at E, are right; wherefore (I. 26) the triangles are everyway equal, and therefore $AE = EB$.



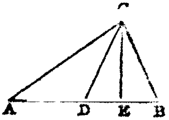
If the point O of intersection of OD, OF, should fall below the base AB, the proof would be exactly similar.

EXERCISE XXIV.—THEOREM.

The angle contained by a line drawn from the vertex of a triangle perpendicular to the base, and another bisecting the vertical angle, is equal to half the difference of the angles at the base.

Let the line CD bisect the vertical angle C of the triangle ABC, and let CE be perpendicular to its base, then angle DCE is = half the difference of angles B and A; or if the side AC be greater than BC, and consequently angle B greater than A, then twice $DCE = B - A$.

In the triangle ACE, having angle E right, it follows (I. 32, Cor.) that the oblique angles A and ACE together are equal to one right angle; for a similar reason angles B and BCE, in triangle BEC, are together equal to a right angle; consequently $A + ACE = B + BCE$.



To these equal quantities, add the angle DCE,

then $A + ACE + DCE = B + BCE + DCE$.

But $ACE = ACD + DCE$ and $BCE + DCE = BCD$,

hence $A + ACD + \text{twice } DCE = B + BCD$.

From these equals take away the equal angles ACD and

BCD, and there remains

$$A + \text{twice } DCE = B;$$

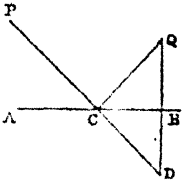
that is, B exceeds A by twice DCE; or, in other words, DCE is equal to half the difference between A and B.

EXERCISE XXV.—PROBLEM.

To find a point in a given line such, that lines drawn from it to two given points will make equal angles with the given line.

Let PQ be the given points, and AB the given line, to find a point as C in AB, so that angles PCA, QCB, may be equal.

Draw QB perpendicular to AB and produce it, making BD = BQ; join PD, and PD will cut AB in the required point C.



Join CQ; then in the triangles BCQ, BCD, the side BD = BQ, and BC is common to the two triangles, and the contained angles CBQ, CBD, are right; hence (I. 4) the triangles are everyway equal, and therefore angle BCQ = BCD. But BCD = PCA (I. 15); wherefore BCQ = PCA, and C is the required point.

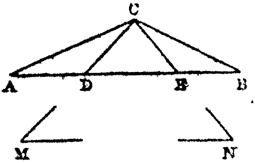
If AB were the face of a plane mirror, and it were required to find the direction in which a ray of light must pass through the point Q, in order that, after reflection, it might pass through P, QC would be that direction; that is, C would be the point of incidence. Were QCP in a horizontal plane, and it were required to find the point at which a perfectly elastic ball proceeding from Q must impinge on a straight obstacle AB, to be reflected to a point P, the same construction is to be used; that is, C is the point. Were the ray or the ball to be reflected at two planes before reaching P, the determination of the two points of incidence is effected in a nearly similar way.

EXERCISE XXVI.—PROBLEM.

Given the sum of the sides of a triangle, and the angles at the base, to construct it.

Let AB be the sum of the sides, and angles M and N the given angles.

At A and B make angles A and B respectively equal to the halves of the given angles M, N, and at C, where the lines AC, BC, meet, draw CD, making angle ACD = A, and CE, making angle ECB = B; then CDE is the required triangle.

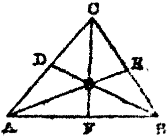


For angle A = ACD, and CDE = A and ACD together; therefore CDE = twice A; but A was made equal to half of one of the given angles; hence CDE = one of the given angles, namely, M; for exactly a similar reason CED = twice B = another of the given angles, namely, N; therefore the angles CDE and CED are respectively equal to the given angles M and N. Again, the side CD = AD, because the opposite angles are equal; for a similar reason CE = EB; consequently CD, DE, and EC, are equal to the whole line AB, which is the given perimeter of the triangle.

EXERCISE XXVII.—THEOREM.

If two lines be drawn from the extremities of the base of a triangle to bisect the opposite sides, the line joining their intersection with the vertex, if produced, will bisect the base.

Let the lines AE, BD, bisect the sides BC, AC, of the triangle ABC; join CO, and CO produced will bisect the base AB in F.



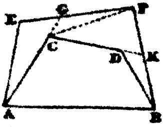
Since BC is bisected in E, the triangles ABE, AEC, are equal (I. 38), as they have the same vertex A; therefore triangle ABE is half of the given triangle ABC; for a similar reason triangle ABD is half of the same; therefore triangle ABE = ABD; and if from these equals the common part AOB be taken away, the remaining triangles AOD, BOE, must be equal. Again, since AC is bisected in D, triangle AOD = COD (I. 38), since they have a common vertex O, and hence triangle AOC is double of AOD; and for a similar reason triangle COB is double of BOE; but AOD = BOE, as was already proved; hence their doubles are equal; that is, triangle AOC = COB; and since these two triangles are on the same base OC, and are equal, they must have the same altitude (I. 37, and A, Scholium 1); that is (Def. 47), the perpendiculars from A and B on CO, or CO produced, must be equal, and consequently the triangles AOF, BOF, have the same altitude, considering OF as their base; and being also on the same base OF, they are therefore equal (I. 37, and A, Scholium 1). Considering now AF, FB, to be the bases of the triangles AOF, BOF, they have the same altitude; as they have O for their common vertex, and as they have been proved equal, they must therefore (I. A) stand on equal bases; consequently AF = FB; that is, the line COF bisects the base.

EXERCISE XXVIII.—THEOREM.

If two polygons be constructed on the same side of the same base, the sum of the sides of the interior polygon, if it be concave internally, is less than the sum of the sides of the exterior figure.

Let ACDB, AEFB, be two polygons, of which the former is internally concave; then its perimeter is less than that of the latter.

For produce AC to G, and CD to K, and join C, F. Then (I. 20) AE + EG is greater than AG; and adding GF + FB to both, AE + EF + FB is greater than AG + GF + FB. Again, CG + GF is greater than CF; and adding AC with FB to both, AG + GF + FB is greater than AC + CF + FB. Also, CF + FK is greater than CK; and adding AC with KB to both, AC + CF + FB is greater than AC + CK + KB. So, DK + KB is greater than DB; and adding AC + CD to both, AC + CK + KB is greater than AC + CD + DB. Hence it has been proved that AE + EF + FB is greater than AG + GF + FB, and this has been proved to be greater than AC + CF + FB, which again was shown to



be greater than $AC + CK + KB$, and this was proved to be greater than $AC + CD + DB$; consequently the first quantity, $AE + EF + FB$, must be still greater than the last, $AC + CD + DB$.

Or more concisely thus, observing that the sign $>$ means greater: by (I. 21)

$AE + EG > AG$, which expression denote by (1);
$CG + GF > CF$, ... (2);
$CF + FK > CK$, ... (3);
and $DK + KB > DB$, ... (4).

To both sides of (1) add $GF + FB$; to those of (2) add $AC + FB$; to those of (3) add $AC + KB$; and to those of (4) add $AC + CD$: then

By (1). $AE + EF + FB > AG + GF + FB$,
... (2). $AG + GF + FB > AC + CF + FB$,
... (3). $AC + CF + FB > AC + CK + KB$,
... (4). $AC + CK + KB > AC + CD + DB$;
consequently $AE + EF + FB$ is still $> AC + CD + DB$.

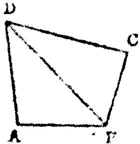
SECOND BOOK.

EXERCISE I.—THEOREM.

The angles of a quadrilateral are equal to four right angles.

Let ABCD be a quadrilateral, its angles at A, B, C, and D, are equal to four right angles.

For draw the diagonal DB, then the angles A, ABD, and ADB, of the triangle DAB, are equal to two right angles; and the angles C, CDB, and DBC, of the triangle DCB, are also equal to two right angles; but the angle ABC is composed of the two angles ABD, DBC, and the angle ADC is composed of the two, ADB, CDB; consequently the angles



A, C, ABC, and ADC, of the quadrilateral, are equal to the angles of the two triangles; that is, to four right angles (I. 32).

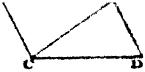
EXERCISE II.—THEOREM.

Every quadrilateral that has its opposite sides, or opposite angles, equal, is a parallelogram.

Let AD be a quadrilateral, having its opposite sides AB, CD, equal, and also AC and BD; or having its opposite angles A and D equal, and also the angles ABD and ACD; then it is a parallelogram.

1. When the opposite sides are equal.

Draw the diagonal BC, then in the two triangles ABC, BCD, the side AB = CD, AC = BD, and BC is common; consequently they are everyway equal (I. 8); hence angle ABC = BCD, and ACB = CBD; therefore AB is parallel to CD, and AC to BD (I. 29), and the figure is a parallelogram (I. Def. 42).



2. When the opposite angles are equal.

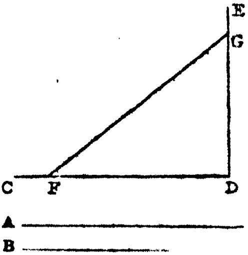
By the last exercise, it was proved that the four angles of a quadrilateral are equal to four right angles; therefore angles A and ABD, which are respectively equal to the angles opposite to them, must be equal to two right angles, and consequently (I. 29) AC and BD are parallel; for a similar reason, angles A and ACD are equal to two right angles, and therefore AB is parallel to CD: therefore ACDB is a parallelogram (I. Def. 42).

EXERCISE III.

To find a line whose square shall be equal to the difference between two given squares.

Let A, B, be the two lines whose squares are the given squares; that is, squares described on them as bases, are the ; to find a line whose square shall be equal to

Take any line CD, draw DE perpendicular to CD, and cut off from it DG = B; and from G as a centre, with a radius = A, cut CD in F; draw GF, and DF is the required line.



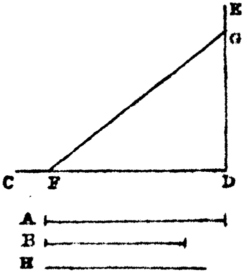
For (I. 47) $FG^2 = DF^2 + DG^2$; or (I. 47, Cor.) $FD^2 = FG^2 - DG^2$, that is, $DF^2 = A^2 - B^2$; or the square of the line DF is equal to the difference between the squares of the lines A and B.

EXERCISE IV.—PROBLEM.

To find a line whose square shall be equal to the sum of the squares of any number of lines.

Let A and B be two given lines; to find a third whose square shall be equal to the squares of these two lines.

Draw any line CD, and cut off from it a part DF = A; make DE perpendicular to CD, and cut off DG = B; join FG, and FG is a line whose square is equal to those of A and B together.



For (I. 47) $FG^2 = DF^2 + DG^2 = A^2 + B^2$.

If three lines are given, as A, B, and H; then find, by the preceding case, a line FG whose square is equal to the sum of the squares of A and B, then, in the same manner, find a line whose square shall be equal to those of FG and the third line H; then it is evident that the square of the line last found is equal to the sum of the squares of A, B, and H.

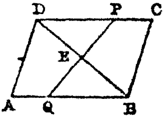
When four lines are given, a fifth line, whose square shall be equal to the sum of the squares of the four given lines, can be found by a similar process.

EXERCISE V.—PROBLEM.

To bisect a parallelogram by a line drawn from a point in one of its sides.

Let ABCD be the parallelogram, and P the point in the side CD, it is required to draw a line, as PQ, to bisect the parallelogram.

Make $BQ = DP$, and draw PQ, and it will bisect the parallelogram.

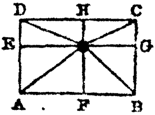


For join DB; then, in the two triangles DEP, BEQ, the vertical angles at E are equal, the alternate angles PDE, QBE, are equal (I. 29), and the sides DP, BQ, are equal by construction; hence the triangles are everyway equal, and therefore their areas are equal. But (I. 34) the two triangles ABD, CBD, are equal; and consequently, if DPE be added to ADB, and EBQ be taken from it, the resulting figure, ADPEQ, will be equal to triangle ABD; that is, equal to half the parallelogram; and the figure BCPQ must be equal to the other half; and the line PQ therefore bisects the parallelogram.

EXERCISE VI.—THEOREM.

If from any point lines be drawn to the angular points of a rectangle, the sums of the squares of those drawn to opposite angles are equal.

Let AC be a rectangle, and O a point in it; draw OA, OB, OC, and OD, then $OA^2 + OC^2 = OD^2 + OB^2$.



For through O draw EG and FH respectively parallel to AB and BC, then EF, FG, GH, and EH, are rectangles, as their opposite sides are parallel, and angles A, B, C, and D, in each, are right angles (I. 46, Cor.) Hence (I. 47) $AO^2 = AF^2 + FO^2 = AF^2 + BG^2$, $OC^2 = OG^2 + GC^2 = FB^2 + GC^2$; therefore, $AO^2 + OC^2 = AF^2 + FB^2 + BG^2 + GC^2$. Again, $OD^2 = DH^2 + OH^2 = AF^2 + CG^2$, and $OB^2 = OF^2 + FB^2 = BG^2 + BF^2$; and therefore, $OD^2 + OB^2 = AF^2 + FB^2 + BG^2 + CG^2$. Hence the sum $AO^2 + OC^2$ and the sum $OD^2 + OB^2$, being each equal to the squares of AF, FB, BG, and GC, must be equal to one another.

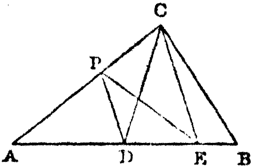
If the point were taken without the rectangle, the proof would be exactly similar.

EXERCISE VII.—PROBLEM.

To bisect a given triangle by a line drawn from a point in one of its sides.

Let ABC be the given triangle, and P the given point in the side AC.

Bisect the base AB in D; join PD, and draw CE parallel to PD; join PE, and PE is the bisecting line required.



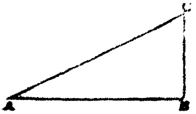
For since CE is parallel to PD, the triangles PCD, PED, are equal (I. 37); to each add ADP, then the sums are equal; that is, the triangle ACD is = APE; but ACD is half of triangle ACB (I. 38), because AD = DB; hence APE is also half of the given triangle, or PE bisects it.

EXERCISE VIII.—THEOREM.

The square of either of the sides of the right angle of a right-angled triangle, is equal to the rectangle contained by the sum and difference of the hypotenuse and the other side.

In the triangle ABC, right-angled at B, the square of AB is equal to the rectangle contained by the sum of AC and CB, and the difference of AC and CB; or $AB^2 = (AC + CB)(AC - CB.)$

For (I. 47, Cor.) $AB^2 = AC^2 - CB^2$, and (II. 5, Cor.) the difference between the squares of AC and CB is equal to the rectangle contained by the sum and difference of AC and CB; that is, $AC^2 - CB^2 = (AC + CB)(AC - CB)$; and consequently

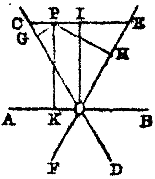


$$AB^2 = (AC + CB)(AC - CB).$$

EXERCISE IX.—THEOREM.

If three straight lines intersecting in the same point make the six angles thus formed equal, the distance of any point from one of the lines is equal to the sum of its distances from the other two.

Let the lines AB, CD, EF, make the six angles about O contained by them equal; then if P be any point, and PK, PG, PH, perpendiculars from it on the sides, then PK is equal to PG and PH together.



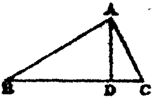
For the angles AOC, COE, EOB, being equal, each of them is the third of two right angles; and if CPE be drawn parallel to AB, angles OCE, OEC, are respectively equal to COA and EOB (I. 29); hence OCE is an equiangular triangle, and therefore equilateral (I. 6, Cor.); hence if OI be perpendicular to AB, IK is a rectangle, and PK = IO. But, as was shown in Ex. 21 of First Book, IO is equal to the perpendicular from E on OC; and OCE being isosceles, and P a point in its base, PG + PH = the latter perpendicular, by Ex. 19 of First Book; hence

$$PG + PH = PK.$$

EXERCISE X.—THEOREM.

The square of the perpendicular upon the hypotenuse of a right-angled triangle drawn from the opposite angle, is equal to the rectangle under the segments of the hypotenuse.

Let ABC be a triangle right-angled at A; then if AD be perpendicular to BC, $AD^2 = BD \cdot DC$.



For $\quad BA^2 = BD^2 + AD^2$ (I. 47),

and $\quad AC^2 = DC^2 + AD^2$;

hence, adding equals to equals,

$$BA^2 + AC^2 = BD^2 + DC^2 + 2AD^2.$$

But $BA^2 + AC^2 = BC^2$ (I. 47) $= BD^2 + DC^2 + 2BD \cdot DC$ (II. 4); wherefore (Ax. 1) $BD^2 +$

$DC^2 + 2AD^2 = BC^2 + DC^2 + 2BD \cdot DC$; and taking away from those equals the common part $BD^2 + DC^2$, there remains,

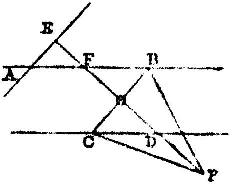
$$2AD^2 = 2BD \cdot DC, \text{ or } AD^2 = BD \cdot DC.$$

EXERCISE XI.—PROBLEM.

Given a point and three lines, two of which are parallel, to find a point in each of the parallels that shall be equidistant from the given point, and such, that the line joining them shall be parallel to the other given line.

Let AB, CD, AE, be the three lines, of which the first two are parallel, and let P be the given point, it is required to find the points B and C such, that BC may be parallel to AE, and that PB may = PC.

From P draw PE perpendicular to AE, and bisect DF in O, and draw COB parallel to AE, then B and C are the required points.

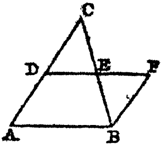


For join PB and PC; then in the two triangles BOF, COD, the sides OD, OF, are equal by construction, the angles at O are equal (I. 15), and the vertical angles BFO, CDO, are also equal (I. 29); hence the triangles are everyway equal, and therefore CO = OB. Again, angle COD = AEF (I. 29), and the angles at E are right by construction; hence COD, and therefore also BOD (I. 13), are right angles, and therefore equal; also the side CO = OB, and OP is common to the two triangles POB, POC; and therefore (I. 4) the triangles are everyway equal; hence PB = PC. The points B, C, are therefore equidistant from P, and the line BC joining them is parallel to AE.

EXERCISE XII.—THEOREM.

The line joining the middle points of two sides of a triangle is parallel to the base, and equal to the half of it.

Let ABC be a triangle, the line DE, that joins the middle points D, E, of two of its sides, is parallel to the third side AB, and equal to its half.



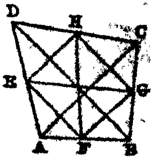
Produce DE to F, making EF = DE, and join BF; then, in the triangles BEF, CDE, the sides DE, EC, are respectively equal to FE, EB, and the vertical angles at E are equal; hence (I. 4) the triangles are equal in every respect, and therefore BF = CD = AD; also angle EBF = C, and hence BF is parallel to CD or DA, and equal to it; therefore (I. 33) DF is parallel and equal to AB. But DE is the half of DF; it is therefore equal to the half of AB, and is also parallel to AB.

EXERCISE XIII.—THEOREM.

The quadrilateral formed by joining the successive middle points of the sides of a given quadrilateral, is a parallelogram.

Let ABCD be any quadrilateral, and EH, HG, GF, FE, lines joining the successive middle points of its sides, then the figure FGHE is a parallelogram.

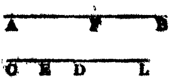
For let AC, DB, be the diagonals of the given figure. Then, by the last exercise, the line EH is parallel to AC, the base of the triangle ACD; for the same reason FG is parallel to AC; consequently FG and EH are parallel (I. 30); and for a similar reason EF and GH are parallel. Hence (I. Def. 42) EFGH is a parallelogram.



EXERCISE XIV.—PROBLEM.

To divide a line into two segments, such that the rectangle contained by one of them and another given line, shall be equal to the square of the other segment.

Let AB, CD, be two lines, it is required to cut AB into two segments in F, so that $AF \cdot CD = BF^2$.



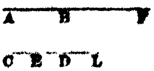
Find (II. 14) a line S (not represented in the figure), such that $S^2 = AB \cdot CD$, this being a given rectangle, since its sides are known. Next bisect CD in E, and find a line EL, such that $EL^2 = CE^2 + S^2$, by Ex. 4 of Second Book; then make $BF = DL$, and F is the required point of section.

For $S^2 = EL^2 - CE^2$, and (II. 5, Cor.) $EL^2 - CE^2 = (EL + CE)(EL - CE) = CL \cdot DL$; for $DL = EL - ED = EL - CE$. Wherefore, since $S^2 = AB \cdot CD$, $AB \cdot CD = CL \cdot DL$; or (II. 1 and 3) $AF \cdot CD + BF \cdot CD = CD \cdot DL + DL^2$; but $BF = DL$, and hence $BF \cdot CD = CD \cdot DL$; and, taking away these equal quantities from the preceding equals, there remains $AF \cdot CD = BF^2$.

EXERCISE XV.—PROBLEM.

To produce a given line, so that the rectangle contained by the former and another given line shall be equal to the square of the produced part.

Let AB, CD, be the given lines, it is required to produce AB to some point F, such that $AF \cdot CD$ may equal BF^2.



As in the preceding exercise, find a line S, such that $S^2 = AB \cdot CD$; bisect CD in E; then find EL, such that $EL^2 = CE^2 + S^2$, and produce AB to F, till $BF = CL$, and F is the required point of external section.

For $S^2 = EL^2 - CE^2 = (EL + CE)(EL - CE)$;

Or, $AB \cdot CD = CL \cdot DL$; for $EL - CE = DL$;

adding respectively to these equals, the equals $BF \cdot CD$, and $CL \cdot CD$; then (II. 1)

$$(AB + BF)\, CD = CL\, (DL + CD);$$

Or, $AF \cdot CD = CL \cdot CL = BF^2$; for $BF = CL$.

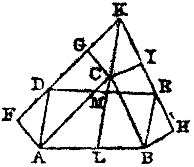
EXERCISE XVI.—THEOREM.

If any parallelogram be described on the base of a triangle, and other two parallelograms on its two sides, such that their sides opposite to those of the triangle shall pass through the angular points of the former, the first parallelo-

gram shall be equal to the sum or difference of the other two, according as they both lie without the triangle, or one of them upon it.

Let ABC be a triangle, and ADEB any parallelogram described on the base, also AFGC, BCIH, parallelograms on the sides, whose sides FG, IH, pass through D, E, angular points of the former, then ABED is = ACGF + BCIH.

Through C draw CL parallel to AD, and produce it to meet FG, HI, produced; these two lines will meet LC in the same point K. For ADKC and BCKE are parallelograms, since their opposite sides are parallel; and hence CK in the former is = AD (I. 34) = BE = CK in the parallelogram CBEK. Since, therefore, the side CK is equal in both, the lines FG, HI, meet LC in one point K. Now (I. 35) the parallelogram ACGF is = ACKD, as they are on the same base AC, and between the same parallels AC, FK; for a similar reason, ACKD is = ALMD, as they are on one base AD, and between the parallels AD, LK; wherefore ACGF is = ALMD. In the same manner it is proved that BCIH is = BLME; and therefore AFGC and BCIH together, are equal to ALMD and BLME together; that is, to the whole parallelogram ABED.

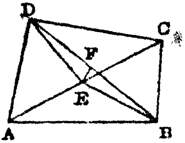


When a side, as AD, of the parallelogram described on the base lies between the corresponding side AC of the triangle and the base AB, the parallelogram on the side AC will then fall on the triangle, and that on the side BC will be equal to the other two. The proof in this case is exactly similar to that in the preceding case.

EXERCISE XVII.—THEOREM.

The sum of the squares of the sides of a quadrilateral is equal to the sum of the squares of its diagonals, and four times the square of the line joining their middle points.

Let ABCD be any quadrilateral, AC, BD, its diagonals, and EF the line joining their middle points, then $AD^2 + DC^2 + CB^2 + BA^2 = AC^2 + DB^2 + 4\,EF^2$.



For draw DE and EB. Then, because the base AC of the triangle ACD is bisected in E, therefore (II. A.)

$$AD^2 + DC^2 = 2\,AE^2 + 2\,DE^2;$$

and $AB^2 + BC^2 = 2\,AE^2 + 2\,BE^2$,

for a similar reason, since AC is also the base of the triangle ABC. Hence, adding equals to equals,

$$AD^2 + DC^2 + AB^2 + BC^2 = 4\,AE^2 + 2\,DE^2 + 2\,EB^2.$$

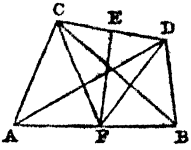
But (II. A.) since BD, the base of triangle DEB, is bisected in F, $2\,DE^2 + 2\,EB^2 = 4\,DF^2 + 4\,EF^2$; consequently the sum of the squares of the sides is equal to $4\,AE^2 + 4\,DF^2 + 4\,EF^2$. But (II. 8, Cor. 2) $4\,AE^2 = AC^2$, $4\,DF^2 = DB^2$; and hence

$$AD^2 + DC^2 + AB^2 + BC^2 = AC^2 + DB^2 + 4\,EF^2.$$

EXERCISE XVIII.—THEOREM.

The sum of the squares of two opposite sides of a quadrilateral, together with four times the square of the line joining their middle points, is equal to the sum of the squares of the other two sides and of the diagonals.

Let ABDC be any quadrilateral, EF the line joining the middle points of the sides AB, CD, and AD, BC, its diagonals, then $AB^2 + CD^2 + 4\,EF^2 = AC^2 + DB^2 + AD^2 + BC^2$.



For join CF and FD; then, because the base AB of the triangles ABC, ABD is bisected in F (II. A.),

$$AC^2 + CB^2 = 2\,AF^2 + 2\,CF^2,$$

and $BD^2 + AD^2 = 2\,AF^2 + 2\,FD^2$;

consequently, adding equals to equals,

$$AC^2 + BD^2 + AD^2 + BC^2 = 4\,AF^2 + 2\,CF^2 + 2\,FD^2.$$

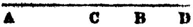
But (II. A.) $2\,CF^2 + 2\,FD^2 = 4\,CE^2 + 4\,EF^2$; wherefore the sum of the squares of AC, BD, AD, and BC, is $= 4\,AF^2 + 4\,CE^2 + 4\,EF^2$.

Again (II. 8, Cor. 2), $4\ AF^2 = AB^2$, and $4\ CE^2 = CD^2$; hence $AC^2 + BD^2 + AD^2 + BC^2 = AB^2 + CD^2 + 4\ EF^2$.

EXERCISE XIX.—THEOREM.

The squares of the sum and of the difference of two lines are together double the squares of these lines.

Let AB, BC, be two lines; produce AB to D, making $BD = BC$; then AD is the sum, and AC the difference of AB and BC, and $AD^2 + AC^2$ is $= 2\ AB^2 + 2\ BC^2$.



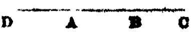
For (II. 4) $AD^2 = AB^2 + BD^2 + 2\ AB \cdot BD = AB^2 + BC^2 + 2\ AB \cdot BC$; for $BD = BC$.

Also (II. 7) $AC^2 + 2\ AB \cdot BC = AB^2 + BC^2$.
And, adding equals to equals,
$AD^2 + AC^2 + 2\ AB \cdot BC = 2\ AB^2 + 2\ BC^2 + 2\ AB \cdot BC$.
Taking away the quantity $2\ AB \cdot BC$ from these equals, there remains, $AD^2 + AC^2 = 2\ AB^2 + 2\ BC^2$.

EXERCISE XX.—THEOREM.

If a line be cut in medial section, the line composed of it and its greater segment is similarly divided.

Let the line AC be cut in medial section at the point B; then, if CA be produced to D, till $AD = AB$, the line CD is also cut in medial section in A.



For $AB^2 = AC \cdot CB$ by hypothesis; and adding to these equals the rectangle $AC \cdot AB$, or $AC \cdot AD$,

$$AC \cdot AB + AB^2 = AC \cdot CB + AC \cdot AB.$$

Or, $AC \cdot AD + AD^2 = AC\ (AB + CB)$.

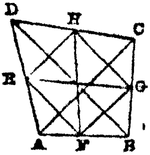
Or (II. 3), $CD \cdot DA = AC^2$ (II. 1).

EXERCISE XXI.—THEOREM.

The sum of the squares of the diagonals of a quadrilateral, is equal to twice the sum of the squares of the lines joining the middle points of the opposite sides.

Let ABCD be any quadrilateral, and EG, FH, lines joining the middle points of the opposite sides, and AC, DB, its diagonals; then

$$AC^2 + DB^2 = 2\,EG^2 + 2\,FH^2.$$



For draw EH, HG, GF, and EF; then, by the 13th Ex., EHGF is a parallelogram; and, by the 12th Ex., DB is double of GH, and AC double of EH; consequently (II. 8, Cor.)

$$AC^2 + DB^2 = 4\,EH^2 + 4\,HG^2.$$

But (I. 34) $4\,EH^2 = 2\,EH^2 + 2\,FG^2$;

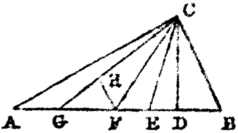
and $4\,HG^2 = 2\,HG^2 + 2\,EF^2$.

Also (II. B.) $EH^2 + HG^2 + GF^2 + EF^2 = EG^2 + FH^2$; and the double of the squares of the sides EH, HG, GF, FE, are $= 2\,EG^2 + 2\,FH^2$, and also to $4\,EH^2 + 4\,HG^2$; hence $AC^2 + DB^2 = 2\,EG^2 + 2\,FH^2$.

EXERCISE XXII.—THEOREM.

If any one of the angles of a triangle be divided into a number of equal parts, the dividing lines will divide the opposite side unequally; the segments nearer to the perpendicular on this side from that angle, being less than the more remote.

Let ABC be a triangle, and CD a perpendicular from angle C on the side AB; let angle ACB be divided into a number of equal parts, of which angles ECF, FCG, are two; then, if EF be nearer to the perpendicular than FG, it will be less than FG.



Since the triangle CDF is right-angled at D, CFD is acute (I. 17); and consequently (I. 13) CFG is obtuse, and therefore greater than CFD. Draw FH, making angle CFH = CFE; there are two angles of triangle EFC equal respectively to two angles in the triangle CFH, and the side CF is common to both; therefore (I. 26) the triangles are every-way equal, and hence FH = FE, and angle CHF = CEF; and consequently (I. 13, Cor.) angle FHG = CED. But angle CED is greater than CGF (I. 16); hence also FHG is greater than FGH; and therefore (I. 19) the side FG of the triangle FGH is greater than FH; but FH was proved to be = FE; therefore FG is greater than FE.

It can easily be proved indirectly, as a corollary from this theorem, that if the divisions EF and FG are equal, the angle ECF, subtended by FE, would be greater than FCG subtended by FG.

THIRD BOOK.

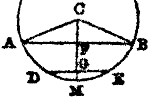
EXERCISE I.—THEOREM.

If a line drawn from the centre of a circle bisect or be perpendicular to a chord, it will bisect and be perpendicular to all chords that are parallel to the former.

Let the line CG, drawn from the centre C of the circle ADEB, bisect the chord AB, or be perpendicular to it, and it will also bisect and be perpendicular to any other chord, as DE, that is parallel to the former.

For if CF bisect AB, it cuts it also at right angles (III. 3); hence AFC is a right angle, and (I. 29) AFC = the interior angle CGD; consequently CG is also perpendicular to DE, and it therefore also bisects DE (III. 3).

Again, if CF be perpendicular to AB, it can be proved as before, that it is also perpendicular to DE, and consequently it also bisects DE.

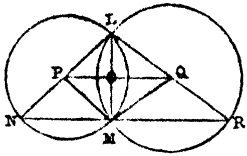


EXERCISE II.—THEOREM.

If two circles cut each other, the line joining the points of intersection is bisected perpendicularly by the line joining their centres.

Let the circles LMN, LMR, intersect in L and M, and let P and Q be their centres, then PQ bisects LM perpendicularly in O.

For draw the radii PL, PM, QL, and QM; then in the two triangles PLQ, PMQ, the side PL = PM, LQ = MQ, and PQ is common; wherefore (I. 8) the triangles are everyway equal, and therefore angle LPQ = MPQ. Again, in the two triangles PLO, PMO, the two sides LP, PO, are respectively equal to MP, PO, and angle LPO = MPO; consequently the triangles are equal in every respect; therefore LO = OM, and angles POL, POM, are also equal, and consequently (I. Def. 10) they are right angles; wherefore LM is bisected perpendicularly by PQ.



EXERCISE III.

If two circles cut one another, and from one of the points of intersection a diameter be drawn through each of the circles, the other point of intersection and the other two extremities of the diameters will be in one straight line.

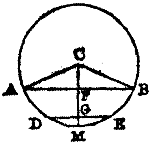
Let the circles LMN, LMR (fig. to preceding Ex.), be two circles intersecting in L and M; and let LN, LR, be two diameters; then N, M, and R, are in one straight line.

For join NM and MR; then the angles LMN and LMR are angles in semicircles; consequently they are right angles, and their sum = two right angles; therefore the lines NM, MR, are in one line (I. 14).

EXERCISE IV.—THEOREM.

A line that bisects two parallel chords in a circle, is also perpendicular to them.

Let the line FG bisect the parallel chords AB, DE; it is perpendicular to them.

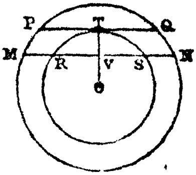


For if CF is a line joining the centre of the circle, and F the middle of AB, it is perpendicular to AB, and (by the first Ex.) if CF be produced, it will bisect DE perpendicularly in G; hence this line must coincide with the line FG, joining the middle points F, G; and consequently FG is perpendicular to the two chords.

EXERCISE V.—THEOREM.

Two concentric circles intercept between them two equal portions of a line cutting them both.

Let PQN, RTS, be two concentric circles, and MN a line cutting them both; then the intercepted portions MR, NS, are equal.

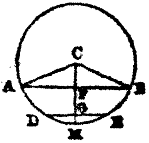


For, from the common centre O, draw OV perpendicular to MN; then MN and RS are bisected in V; hence $MV = VN$, and $RV = VS$; and therefore $MV - RV = VN - VS$; that is, $MR = NS$.

EXERCISE VI.—THEOREM.

Of these five conditions—of passing through the centre of a circle, bisecting a chord, being perpendicular to a chord, bisecting the subtended angle at the centre, bisecting the subtended arc of the chord—if a line fulfil any two, it will also fulfil the other three.

Let AMB be a circle, AB a chord, and CF a line bisecting AB, or cutting it at right angles, or bisecting the arc AME.



1. If CF bisect AB, it cuts AB perpendicularly (III. 3).

2. If CF cut AB perpendicularly, it bisects AB (III. 3).

3. If CF bisect AB perpendicularly, it passes through the centre (III. 3, Cor.)

4. If CF bisect AB, or cut AB at right angles, it will bisect angle ACB, and also the arc AMB.

For (III. 3) the triangles ACF, BCF, are everyway equal in both these cases, and hence angle ACM = BCM; and therefore (III. 26) the arc AM = MB.

5. If CM bisect angle ACB, or the arc ACB, it will also bisect AB, and cut it at right angles.

For if angles ACF, BCF, are equal, then since AC = CB and CF, common to the triangles ACF, BCF, hence (I. 4) the triangles are everyway equal, and therefore AF = FB, and angle AFC = BFC, and hence they are right angles.

Also, when angle ACB is bisected, the arc AM = MB, for they subtend equal angles at the centre (III. 27).

Again, if CM bisects the arc AMB, the angles ACM, BCM, stand on equal arcs, and are therefore equal; and therefore it can be proved, as above, that CM also bisects AB, and is therefore perpendicular to it.

6. If the line FM bisect the arc AMB, and also AB, it will cut AB at right angles; it will pass through the centre C, and will bisect the angle ACB.

For it has been proved that a line CM, drawn from the centre to M, the middle of the arc, bisects AB; hence the line MF, drawn from M to F, must coincide with the former, and consequently will pass through the centre, will also cut AB at right angles, and bisect angle ACB.

7. If the line FM bisect the arc AMB, and cut AB at right angles, it will bisect AB, will pass through the centre, and will bisect the angle ACB.

This case can be proved exactly in the same manner as the preceding.

8. If CM bisect the arc AMB, and also the angle ACB, it will bisect AB, and be perpendicular to AB, and will evidently pass through the centre C.

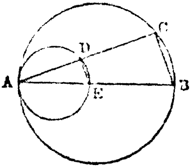
Since the line must in this case pass through C, and as it also bisects the angle ACB, this case is thus reduced to the first part of the fifth case above.

The fourth case above properly contains two cases, and so does the fifth; there are thus, in all, ten cases.

EXERCISE VII.—THEOREM.

If a circle be described on the radius of another circle, any chord in the latter, drawn from the point in which the circles meet, is bisected by the former.

Let ABC be a circle, AE a radius of it, and ADE a circle described on AE as a diameter; then any chord, as AC, in the circle ABC, is bisected in D.



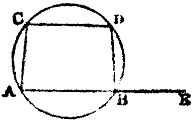
For join DE; then angle ADE in a semicircle is a right angle (III. 31), and consequently (III. 3) AC is bisected in D.

EXERCISE VIII.—THEOREM.

The exterior angle of a quadrilateral figure inscribed in a circle is equal to the interior and opposite.

Let ABDC be a quadrilateral inscribed in a circle, its exterior angle DBE is equal to its interior opposite angle ACD.

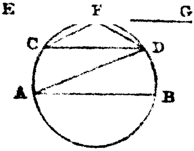
For (III. 22) the opposite angles ACD, ABD, together, are equal to two right angles, also (I. 13) ABD and DBE are together equal to two right angles; consequently the first pair of angles are equal to the second pair; and taking from both the common angle ABD, there remains angle ACD = DBE.



EXERCISE IX.—THEOREM.

Parallel chords in a circle intercept equal arcs.

Let AB, CD, be two chords in a circle; then the intercepted arcs AC, BD, are equal.

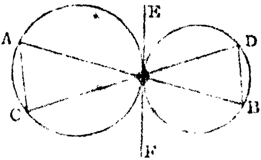


For join AD; then (I. 29) the alternate angles ADC, BAD, are equal, and therefore (I. 26) the arcs AC, BD, on which the angles stand, are equal.

EXERCISE X.—THEOREM.

If two circles touch one another, either internally or externally, the chords of the two arcs, intercepted between two lines drawn through the point of contact, are parallel.

Let ACO, BDO, be two circles touching in O; AB, CD, any two lines through O; then the chords AC, DB, are parallel.



For draw the line EF through O, and perpendicular to the line joining the centres of the circles, which passes through the point of contact (III. 12); hence (III. 16) EF is a tangent to both circles; consequently (III. 32) angle A = COF, and (I. 15) COF = DOE, and (III. 32) DOE = B; wherefore angle A = B, and AC is parallel to DB (I. 27).

EXERCISE XI.—THEOREM.

If a tangent to a circle be parallel to a chord, the point of contact bisects the intercepted arc.

Let the tangent EG and chord CD to the circle ABF (fig. to 9th Ex.) be parallel, then the arc CFD is bisected in F.

For since EG is a tangent, therefore (III. 32) angle CFE = CDF; and because EG is parallel to CD, angle CFE = FCD; consequently angle CDF = FCD; and therefore (III. 26) the arc CF is = DF.

EXERCISE XII.—THEOREM.

If a chord to the greater of two concentric circles be a tangent to the less, it is bisected in the point of contact.

Let MPN, RTS (fig. to the 5th Ex.) be two concentric circles, and PQ a chord to the greater, touching the less in T, then PT = TQ.

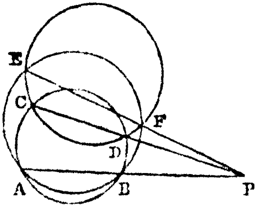
For if OT be drawn, it will be perpendicular to PQ; and since O is the centre of the circle MPN, and OT perpendicular to the chord PQ, it bisects the chord (III. 3); hence PT = TQ.

EXERCISE XIII.—THEOREM.

If any number of circles intersect a given circle, and pass through two given points, the lines joining the intersections of each circle will all meet in the same point.

Let A, B, be two given points, and CDFE a given circle; then if any number of circles ABC, ABE, be described through A, B, cutting the circle CDFE, then the lines CD, EF, produced, will meet AB in the same point P.

For let CD meet AB, when both are produced, in the point P, and from P draw a line PFE, cutting the circle CDFE in E and F; then the circle described through A, B, and E, will cut PE in F, the point in which it cuts CDEF. For if not, let it cut PE in another point F′ (not shown in the figure); then as PC, PE, are secants of the circle CDEF, therefore (III. 37) PE · PF = PC · PD = PA · PB; but PA, PF′E, being secants of the circle ABEF′, therefore PA · PB = PE · PF′; hence PE · PF = PE · PF′; and consequently PF = PF′, or F′ coincides with F. Wherefore the line joining the points of section E, F, meets AB in the same point P in which CD meets it; and the same can be proved of any other circle passing through A and B, and cutting the circle CDEF.



COR.—Hence, if the rectangles PA · PB and PE · PF are equal, a circle can be described through A, B, E, and F.

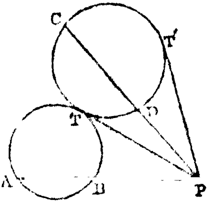
EXERCISE XIV.—PROBLEM.

Through two given points, to describe a circle touching a given circle.

Let A, B, be the given points, and CDT the given circle, to describe a circle through A and B, to touch CDT.

Let any circle passing through A and B cut the given circle in C and D; draw CD and AB, and produce them to meet in P; from P draw the tangents PT, PT′, to the given circle CTD; then if a circle be described passing through A, B, and T, or through A, B, and T′, it will touch the given circle either externally or internally.

For since PT is a tangent to the given circle CTD, therefore $PT^2 = PC \cdot PD$ (III. 36); but A, B, C, D, being, by hypothesis, points in the circumference of a circle, hence $PC \cdot PD = PA \cdot PB$; wherefore $PA \cdot PB = PT^2$, and consequently (III. 37) PT is also a tangent to the circle ABT. Since PT is a common tangent, the circles must touch in T; for the tangent lies between them, so that they can meet only in the point T.



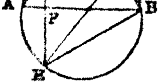
If a circle be described through A, B, and T′, it could be similarly proved that it would touch the given circle in T′.

EXERCISE XV.—THEOREM.

If two chords in a circle intersect each other perpendicularly, the sum of the squares of their four segments is equal to the square of the diameter.

Let ABDC be a circle, and AB, CE, two perpendicular chords in it, then the sum of the squares of the segments AF, FC, EF, FB, is equal to the square of the diameter.

For draw the diameter ED, and join CD, AC, DB, and EB. Then angle ECD is right (III. 31), and therefore = EFB; consequently (I. 29) CD is parallel to AB; and hence (9th Ex.) the arc AC = DB; wherefore (III. 26) the chord AC = DB. Also angle EBD in a semicircle is right (III. 31). Now (I. 47) $AF^2 + FC^2 = AC^2 = DB^2$; and $EF^2 + FB^2 = EB^2$; also $EB^2 + DB^2 = DE^2$; conse-

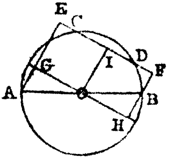


quently, adding equals to equals, $AF^2 + FC^2 + EF^2 + FB^2 = DE^2$.

EXERCISE XVI.—THEOREM.

Perpendiculars from the extremities of a diameter of a circle, upon any chord, cut off equal segments.

Let ABDC be a circle, AB a diameter, CD a chord, and AE, BF, perpendiculars on the chord; then the segment CE = DF.

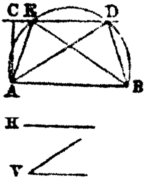


For draw GH through the centre O, parallel to CD, and OI perpendicular to EF. Then because AE, OI, are perpendicular to EF, they are parallel (I. 28); and EI, GO, are parallel by construction; consequently GI is a parallelogram, and it is rectangular (I. 46, Cor.) It is similarly shown that IH is a rectangle; wherefore GO = EI, and OH = IF. Now, in the triangles AOG, BOH, the vertical angles at O are equal, the alternate angles at A and B, and the side AO = OB; consequently (I. 26) the triangles are everyway equal; hence GO = OH; wherefore EI = IF, and (III. 3) CI = ID; therefore the remainder CE = DF.

EXERCISE XVII.—PROBLEM.

Given the vertical angle, the base and altitude of a triangle, to construct it.

Let AB be the base, V the vertical angle, and H the altitude of the triangle, to construct it.



On the base AB describe a segment of a circle ABDE, containing an angle equal to the given angle; and draw AC perpendicular to AB, and make AC = H; through C draw CD parallel to the base, and let it cut the arc in D and E; join AD and DB, and ADB is the required triangle.

For the vertex of the triangle must lie in the line CD (I. 34, Cor.); it must also evidently lie in the arc of the segment (III. 21); hence it must be either of the points D or E. Join AD and DB, and ADB is one of the required triangles; for it has the given base, the given height, and vertical angle.

By joining AE and BE, it can be similarly shown that the triangle ABE fulfils the three required conditions of having a base, altitude, and vertical angle of the given magnitudes.

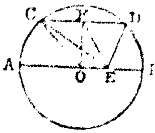
The two triangles ADB, ABE, are evidently equal in every respect, though they are in reverse positions. They could not be made to coincide, without first conceiving one of them to be turned round some line, as one of its sides, as an axis, till it would describe half a revolution, and thus again come into the same plane, though in a reverse position to its present.

EXERCISE XVIII.—THEOREM.

In a circle, the sum of the squares of two lines drawn from the extremities of a chord, to any point in a diameter parallel to it, is equal to the sum of the squares of the segments of the diameter.

Let ABDC be a circle, of which AB, CD, are a diameter and a chord parallel to it; then if E be any point in the diameter, $CE^2 + ED^2 = AE^2 + EB^2$.

For draw OF perpendicular to CD, and join EF and OC. Then (III. 3) CD is bisected in F; OF is perpendicular to AB (I. 29); and hence (II. A.) $CE^2 + ED^2 = 2\,CF^2 + 2\,EF^2$. Again, $2\,EF^2 = 2\,OE^2 + 2\,OF^2$ (I. 47); consequently $2\,CF^2 + 2\,EF^2 = 2\,CF^2 + 2\,OF^2 + 2\,OE^2$. But $2\,CF^2 + 2\,OF^2 = 2\,OC^2 = 2\,AO^2$ (I. 47); therefore $2\,CF^2 + 2\,EF^2 = 2\,AO^2 + 2\,OE^2 = AE^2 + EB^2$ (II. 9). Hence $CE^2 + ED^2 = AE^2 + EB^2$.

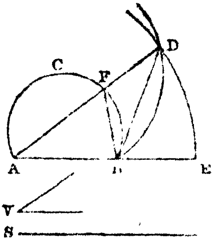


EXERCISE XIX.—PROBLEM.

Given the vertical angle, the base, and the sum of the sides of a triangle, to construct it.

Let AB be the base, V the vertical angle, and S the sum of the sides, to construct the triangle.

On the base AB describe a segment of a circle ACB, containing an angle equal to V, and another arc, of which BD is a portion, containing an angle $= \frac{1}{2}$ V; then with a radius AE = S, and centre A cut BD in D; join AD and BF, and ABF is the required triangle.



For join BD; then angle AFB = twice D by construction; but AFB = D + DBF (I. 32); hence D + DBF = 2 D; and therefore DBF = D, and consequently DF = BF. But AF + FD = AD = AE = S; therefore also AF + FB = S. The triangle ABF, therefore, has the given base, the given sum of the sides, and the given vertical angle; it is, therefore, the required triangle.

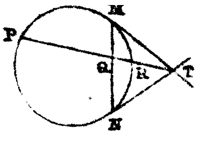
Had the difference of the sides been given instead of the sum, then, instead of describing the segment of which BD is a part, it would have been necessary to describe a segment on AB containing an angle equal to V, added to half its defect from two right angles; that is, to V + $\frac{1}{2}$ BFD; and then from A as centre, with a radius equal to the difference, to cut this latter arc in a point, say D; then the remaining construction and proof would have been analogous to that above.

EXERCISE XX.—THEOREM.

If the points of contact of two tangents to a circle be joined, any secant drawn from their intersection is divided into three segments such, that the rectangle under the secant and its middle segment, is equal to that under its extreme segments.

Let PMN be a circle, TM, TN, two tangents, MN a chord joining the points of contact, and PT a secant; then

$$PT \cdot QR = PQ \cdot RT.$$



For (III. 37, Cor.) MT = TN, and hence MNT is an isosceles triangle; consequently (II. D.)

$MT^2 = QT^2 + MQ \cdot QN = QT^2 + PQ \cdot QR$ (III. 35);
but $MT^2 = PT \cdot TR$ (III. 36) $= PQ \cdot TR + QT \cdot TR$;
hence $QT^2 + PQ \cdot QR = PQ \cdot TR + QT \cdot TR$.
Or, since $QT^2 = QT \cdot QR + QT \cdot TR$ (II. 2);
therefore $QT \cdot QR + QT \cdot TR + PQ \cdot QR = PQ \cdot TR + QT \cdot TR$;
and taking $QT \cdot TR$ from both these equals, there remains

$$QT \cdot QR + PQ \cdot QR = PQ \cdot RT.$$

Or (II. 1), $PT \cdot QR = PQ \cdot RT$.

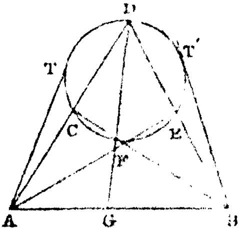
EXERCISE XXI.—THEOREM.

If the opposite sides of a quadrilateral inscribed in a circle be produced to meet, the square of the line joining the points of concourse is equal to the sum of the squares of the two tangents from these points.

Let CDEF be a quadrilateral inscribed in a circle, and let the opposite sides produced meet in A and B, then

$$AB^2 = T^2 + T'^2,$$

where T, T', denote the lengths of the tangents AT, BT', drawn from A and B to the circle.



Divide AB in G, so that $AB \cdot BG = T'^2$. This can be done by drawing from B a secant to the circle CDEF equal to BA; then its external segment will be BG. Then since $T'^2 = BD \cdot BE$ (III. 36), hence $AB \cdot BG = DB \cdot BE$, and consequently a circle may be described through the points A, G, E, and F; hence angle AGD = DEF (III. 22); wherefore (I. 13, Cor., and III. 22) since angle AGD + BGD = two right angles, and DEF + DCF = two right angles, therefore angle BGD = BCD, and consequently a circle may be described through the points B, G, C, and D; and consequently $AB \cdot AG = AD \cdot AC = T^2$; but $AB \cdot BG = T'^2$, whence $AB \cdot AG + AB \cdot BG = T^2 + T'^2$.

Or (II. 1), $AB\,(AG + BG) = AB^2 = T^2 + T'^2$.

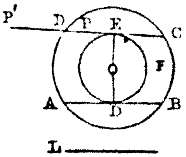
FOURTH BOOK.

EXERCISE I.—PROBLEM.

Through a given point within or without a circle, to draw a chord that shall be equal to a given line.

Let ABCD be a circle, P a point within it, and L a given line, to draw through P a chord that shall be = L.

Place any chord AB in the circle = L, draw from the centre O the perpendicular OD to AB, and describe with the radius OD the circle DEF concentric with the given circle; from P draw the tangent PE to the interior circle, and produce it to C and D, then CD is the required chord.



For join O, E, then OE is perpendicular to CD (III. 16); consequently (III. 15) since OE = OD, the chord CD is = AB, and AB = L; therefore CD = L.

If the point were without the circle, as P′, the construction would be the same as in the preceding case.

EXERCISE II.—PROBLEM.

To draw a chord in a circle that shall be equal to a given line, and parallel to another given line, or inclined to it at a given angle.

Let L be the given line, and ABCD the given circle (fig. to preceding Ex).

As in the last Exercise, place a line AB in the circle ABCD = L, and draw OD from the centre O perpendicular to AB; then if the required chord is to be parallel to L, draw from the centre O a line OE, in a direction perpendicular to L, and make OE = OD; then draw a chord DEC through E, perpendicular to OE, and it is the required chord.

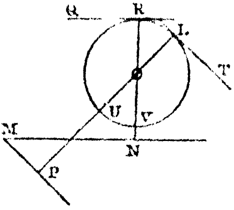
For since $OE = OD$, therefore (III. 14) $CD = AB$; but AB was made $= L$; hence $CD = L$. And because OE is perpendicular to L and CD, therefore these two lines are parallel.

Again, if the required chord is to be inclined to L at a given angle, then draw a line M (not shown in the figure) through any point in L, making the given angle with L; then, as in the former case, draw a chord equal to L, and parallel to M, and it would evidently be the required chord.

EXERCISE III.—PROBLEM.

To draw a tangent to a given circle, so that it shall be parallel to a given line.

Let RLVU be the given circle, and MN the given line, to draw a tangent to the circle parallel to MN.



Draw a line ON from the centre perpendicular to MN, and produce the line to R; then at R draw QR perpendicular to OR, and it is the required tangent.

For RN being perpendicular to both RQ and MN, the alternate angles at R and N are equal; and hence (I. 27) QR is parallel to MN.

EXERCISE IV.—PROBLEM.

To draw a tangent to a circle, so that it shall make a given angle with a given line.

Let RLVU be the given circle, and MN the given line (fig. to 3d Ex).

Draw MP from any point M in MN, so as to be inclined to MN at an angle $M =$ the given angle; then, by the preceding Exercise, draw a tangent LT parallel to MP, and it is the required tangent.

For if LT be produced to meet MN in some point W (not shown in the figure), the angle W would be equal to the alternate angle M; hence LT is inclined to MN at the given angle.

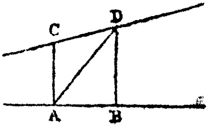
EXERCISE V.—PROBLEM.

To find a point in a given line that shall be equidistant from another given point in it, and from another given line.

Let AB, CD, be two given lines, and D a point in the latter, to find another point, C, in it that shall be equidistant from D and from AB.

From D draw DB perpendicular to AB, and bisect angle CDB by DA; let DA produced cut AB in A; then draw AC parallel to BD, and C is the required point.

For the alternate angles CAD, ADB, are equal, and ADB is equal to ADC by construction; hence also CAD = ADC, and consequently (I. 5) AC = CD, and C is the required point.



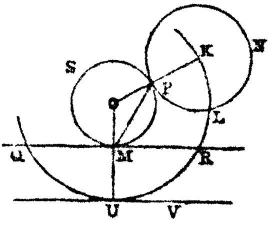
EXERCISE VI.—PROBLEM.

To describe a circle that shall touch a given line in a given point, and pass through another given point.

Let QR be the given line, and P the given point, to describe a circle passing through P, and touching QR in M.

Draw MO perpendicular to QR, join MP, and draw PO, making angle P = OMP, then from O as a centre, with the radius OM, describe the circle MPS, and it is the required circle.

For since angle P = OMP, therefore OP = OM, and a circle described from the centre O, with the radius OM, will pass through the points M and P; and since OM is perpendicular to QR, the circle also touches QR.

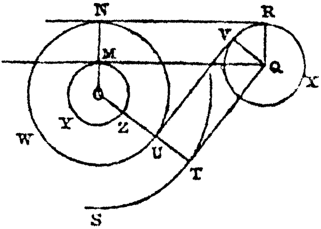


EXERCISE VII.—PROBLEM.

To draw a line that shall be a common tangent to two circles.

Let NUW, RVX, be the two circles, it is required to draw a common tangent to them.

About the centre O, describe a circle MYZ, whose radius OM is the difference between those of the given circles; that is, OM = ON — RQ; from the centre Q draw MQ, a tangent to MYZ; join OM, and produce OM to N; draw RQ parallel to MN, and join NR; then NR is the required tangent.



For OM = ON — RQ, but OM = ON — MN; hence MN must = RQ, and is also parallel to it; wherefore NR is equal and parallel to MQ (I. 33), and MR is a rectangle (I. Def. 40, and I. 46, Cor.); whence angles N and R are right, and therefore NR is a common tangent (III. 16).

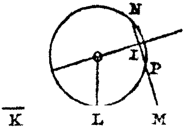
If from centre O, with a radius OT = the sum of ON and RQ, an arc ST be described, and a tangent to it, QT, be drawn from the centre Q, and QV be drawn parallel to OT, and UV joined, it can be shown, in exactly the same way as in the preceding case, that UQ is a rectangle, and UV a common tangent.

EXERCISE VIII.—PROBLEM.

To find a point in a given line that shall be equidistant from another given point and a given line.

Let P be the given point, and KM, OI, the given lines, to find a point O in OI that shall be equidistant from P and KM.

Draw PI perpendicular to OI, and produce it to N and P, and let IN = IP; find ML, the side of a square equal to the rectangle $MN \cdot MP$ (II. 14); through L draw LO perpendicular to KM, and O is the required point.



For if a circle LPN be described through the three points LPN, then since $MN \cdot MP = ML^2$, MK is a tangent to the circle (III. 37); consequently the centre is in LO (III. 19); also, since IO bisects the chord PN perpendicularly, the centre of the circle is in IO (III. 3, Cor.); consequently O must be the centre, and therefore OL = OP; and O is the required point.

EXERCISE IX.—PROBLEM.

To describe a circle that shall pass through two given points, and touch a given line.

Let P, N, be the given points, and KM the given line, to describe a circle passing through P and N, and touching KM (fig. to preceding Ex).

Join N, P, and bisect it perpendicularly by IO; then, as in the last problem, find O the centre of the circle LPN, and describe it, and it is the required circle.

EXERCISE X.—PROBLEM.

To describe a circle that shall touch a given line in a given point, and also touch a given circle.

Let PLN be the given circle, and QMR the given line, and M the given point in it, to describe a circle touching PLN, and also QR in M.

Draw OMU perpendicular to QR (fig. to 6th Ex.), and a line VU parallel to QR, and at a distance from it, equal to PK, the radius of PLN. Let K be the centre of PLN, and describe a circle (by 6th Ex.), KLU, passing through K, and touching UV in U; then since OM is perpendicular to QR, it is so to UV, for angle OMR = OUV (I. 29); hence the centre of the circle KLU must lie in OU (III. 19), and its centre O is the centre of the required circle PMS, of which OM or OP is the radius.

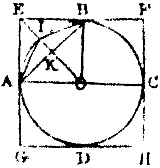
For $OU = OK$, and $MU = PK$; hence $OM = OP$; and therefore a circle described from O as a centre, with the radius OM, will pass through M and P; and since OK passes through P, and a perpendicular to OK at P would be a tangent to both circles, the circles must touch in P; also angle OMR being a right angle, the circle PMS touches the line QMR.

EXERCISE XI.—THEOREM.

A regular octagon inscribed in a circle is equal to the rectangle under the sides of the inscribed and circumscribing square.

Let ABCD be a circle, AB a side of the inscribed square, EFHG the circumscribed square, and AI, IB, two sides of the inscribed octagon; then the area of the octagon $= AB \cdot AC$.

For (IV. 6 and 7) the diameter AC, and the radius BO, are mutually perpendicular, also AC is parallel to EF and GH, and equal to the sides of the circumscribed square; also the chords AI, IB, being sides of the octagon, are equal, and hence the arc AI = IB (III. 28); and consequently (6th Ex. of Third Book) OI is perpendicular to AB, and bisects it. Hence,



twice triangle $OBI = IO \cdot BK$ (I. 41); also
twice triangle $OAI = IO \cdot AK$; and therefore
twice $AOBI = IO\,(AK + BK) = IO \cdot AB$ (II. 1);
Or, $4\,AOBI = 2\,AO \cdot AB = AC \cdot AB$;
and the octagon is evidently four times the quadrilateral AOBI; hence it also $= AC \cdot AB$.

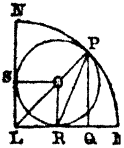
EXERCISE XII.—PROBLEM.

To inscribe a circle in a given quadrant.

Let LMN be the quadrant, it is required to describe a circle in it.

Draw LP, bisecting the right angle MLN; from P draw PQ parallel to LN, and PR, bisecting angle LPQ; then from R draw RO parallel to LN, to meet LP in O, and O is the centre of the circle required, and OP or OR its radius.

For draw OS parallel to LM. Then, since PLM is half a right angle, and ORL is a right angle (I. 29), hence LOR is also half a right angle, and therefore $LR = RO$; but $LR = OS$ (I. 34), whence $RO = OS$. Again, angle $OPR = QPR$ by construction, and $QPR = PRO$, being alternate (I. 29); therefore $OPR = PRO$, and hence $OR = OP$; wherefore OR, OP, OS, are all equal; and a circle being described, with any of these lines as radius from the centre O, it will pass through the points P, R, S. It will also touch the radii LM, LN, because the angles at R and S are right, since SR is a rectangle (I. 46, Cor.); and it touches the arc MPN in P, for the line OL, joining the centres, passes through P, and a perpendicular to LP at P would be a tangent to the quadrantal arc and the circle PRS.

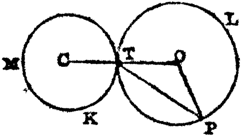


EXERCISE XIII.—PROBLEM.

To describe a circle that shall pass through a given point, and touch a given circle in a given point.

Let KMT be the given circle, and T the given point of contact, and P another given point; to describe a circle passing through P, and touching KMT in T.

Through T draw the radius CT from the centre C, and produce CT; join P and T, and make angle $P = OTP$, then O is the centre of the required circle; and if from O as centre, with a radius $= OP$, a circle, PLT, be described, it will pass through P, and touch the circle KMT in T.



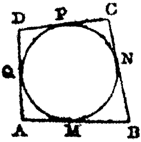
For angle $P = OTP$; hence $OT = OP$, and the circle touches MKT in T, because the circles meet in T, and a line perpendicular to OC at T would be a common tangent to them (III. 16).

EXERCISE XIV.—THEOREM.

If a quadrilateral circumscribe a circle, any two of its opposite sides are together equal to half its perimeter.

Let ABCD be a quadrilateral circumscribing the circle MNPQ, then $AB + CD$, or $AD + BC$, is $=$ to the half of $AB + BC + CD + AD$.

For (III. 37, Cor.) the tangent $AM = AQ$, $BM = BN$, $DP = DQ$, and $CP = CN$; hence the sum of the first of these four pairs of equal quantities is equal to the sum of the other four; that is, $AM + BM + DP + CP = AQ + BN + DQ + CN$.



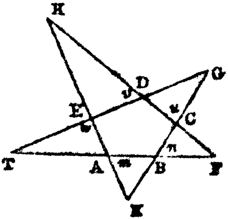
Or, $AB + CD = AD + BC$; that is, the sum of two opposite sides is equal to the sum of the other two sides; wherefore the sum of either pair of opposite sides is equal to half the perimeter.

Cor.—The sum of one pair of opposite sides of a quadrilateral circumscribing a circle, is equal to the sum of the other pair of sides.

EXERCISE XV.—THEOREM.

If every two alternate sides of a polygon be produced to meet, the sum of the salient angles thus formed, with eight right angles, will be equal to twice as many right angles as the figure has sides.

Let ABCDE be a polygon, and let its sides produced meet in F, G, H, T, K; then the sum of the salient angles F, G, H, T, and K, with eight right angles, is equal to the interior angles of the figure.



The number of triangles exterior to the figure is evidently equal to the number of sides, and the sum of their angles therefore equal to two right angles repeated as often as the figure has sides

(I. 32). But any angle, as *m*, at the base of one of these triangles, is equal to the opposite angle TAE in the contiguous triangle AET; hence, since the angles *m*, *n*, *u*, *v*, *w*, are exterior angles of the polygon, they are equal to four right angles, and consequently all the angles at the bases of the triangles are together equal to eight right angles; wherefore the salient angles F, G, H, T, K, with twice the angles *m*, *n*, *u*, *v*, *w*, that is, with eight right angles, are equal to twice as many right angles as the figure has sides.

Cor.—The salient angles are equal to the interior angles with four right angles; and therefore the salient angles, with four right angles, are equal to the interior angles of the figure.

The proof will appear more clear and concise thus :—

Let S denote the salient angles,
B the angles at the bases of the triangles,
I the interior angles of the polygon,
R a right angle, and
n the number of the polygon's sides ;

then since the exterior angles m, n, u, v, w, are equal to their vertically opposite angles, all the angles at the bases; that is,

$$B = 8 \text{ right angles.}$$

But $S + B = 2\,R$ repeated as often as there are triangles, or $S + 8\,R = 2\,R \times n$.

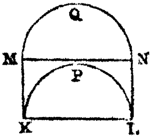
Again, $I + 4\,R = 2\,R \times n$ (I. 32, Cor. 1);
hence $S + 8\,R = I + 4\,R$;
and taking away 4 R from these equals,
there remains $S + 4\,R = I$.

EXERCISE XVI.—THEOREM.

If on two opposite sides of a rectangle semicircles be described lying on corresponding sides of their diameters, the mixtilineal space contained by their arcs, and the other two sides of the rectangle, is equal to the rectangle.

Let ML be the rectangle, KPL, MQN, semicircles described on its opposite sides KL, MN; then the mixtilineal space KMQNLP is equal to the rectangle.

For from the whole figure KMQNLK take away the semicircle MQN, and there remains the rectangle KMNL; and from the same figure take away the other semicircle KPL, and there remains the mixtilineal space KMQNLP; and since the semicircles are equal, as their diameters are so, therefore the remainders are equal; that is, the given rectangle ML is equal to the mixtilineal space KMQNLP.



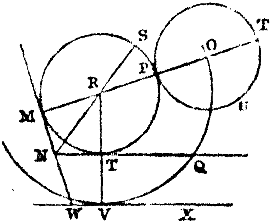
EXERCISE XVII.—PROBLEM.

To describe a circle that shall touch two given lines, and pass through a given point.

Let MN, NQ, be the two given lines, and P the given point; to describe a circle through P, and touching MN and NQ.

Draw NS bisecting angle MNQ (I. 9), then, by the 8th Exercise, find a point, R, in NS equidistant from P and the line NQ (by the 8th Ex.); and R is the centre of the required circle.

For (by the 8th Ex. in First Book) R is equidistant from MN and NQ, and also from P and NQ; hence RT, RM, RP, are equal, and a circle described from the centre R, with either of these three lines as radius, will pass through P, T, and M. The circle will also touch MN, NQ, for the angles at M and T are right angles.



EXERCISE XVIII.—PROBLEM.

To describe a circle that shall touch two given lines and a given circle.

Let MN, NQ, be the given lines, and PTU the given circle; to describe a circle touching the given lines and the given circle. (See the fig. to last Ex.)

Draw a line WX at a distance from NQ, equal to the radius OP of the given circle, and parallel to NQ; bisect angle MNT by NS, and, as in the preceding Exercise, find a point R in NS, the centre of a circle passing through O, the centre of the given circle, and touching WX in V; then R is the centre of the required circle. For draw the radius RV; then, as in the 10th Exercise, it can be easily proved that, since RV = RO, TV = PO, and therefore RT = RP, the circle described from R as a centre, with either RT or RP as a radius, will touch the circle PTU in P, the line NQ in T, and consequently (by the preceding Ex.) it will also touch NM; hence MPT is the required circle.

FIFTH BOOK.

EXERCISE I.—THEOREM.

If all the terms, or any two homologous terms, or the terms of either of the ratios of an analogy, be multiplied or divided by the same number, the resulting magnitudes are still proportional.

Let $A : B = C : D$, and m, n, any two numbers; then,

First, $mA : mB = mC : mD$.

Second. $\frac{1}{m}A : \frac{1}{m}B = \frac{1}{m}C : \frac{1}{m}D$.

Third, $mA : B = mC : D$.

Fourth, $\frac{1}{m}A : B = \frac{1}{m}C : D$.

Fifth, $mA : mB = C : D$.

Sixth, $\frac{1}{m}A : \frac{1}{m}B = C : D$.

First. Since $A : B = mA : mB$, and $C : D = mC : mD$ (V. 15), hence, from equal ratios (V. 11), $mA : mB = mC : mD$.

Second. Since A and B are multiples of $\frac{1}{m}A$ and $\frac{1}{m}B$ by m, therefore $A : B = \frac{1}{m}A : \frac{1}{m}B$ (V. 15); similarly, $C : D = \frac{1}{m}C : \frac{1}{m}D$; consequently (V. 11) $\frac{1}{m}A : \frac{1}{m}B = \frac{1}{m}C : \frac{1}{m}D$.

Third. Since $A : B = C : D$, therefore (V. 4, Cor.) $mA : B = mC : D$.

It is similarly proved that $A : nB = C : nD$; for this is the case of $m = 1$ in V. 4; as in the corollary in the case of $n = 1$.

Fourth. Since $A : B = C : D$, by alternation $A . C = B : D$; and therefore, as in the second case above, $\frac{1}{m}A : \frac{1}{m}C = B : D$; and again, by alternation, $\frac{1}{m}A : B = \frac{1}{m}C : D$.

Fifth. Since $A : B = C : D$, and (V. 15) $A : B = mA : mB$, therefore, by equal ratios (V. 11), $mA : mB = C : D$.

Sixth. Since $A : B = C : D$, and since A, B, are equimultiples of $\frac{1}{m}A$, $\frac{1}{m}B$, by m; therefore (V. 15) $A : B = \frac{1}{m}A : \frac{1}{m}B$; and consequently (V. 11) $\frac{1}{m}A : \frac{1}{m}B = C : D$.

EXERCISE II.—THEOREM.

If any number of magnitudes be in continued proportion, the difference between the first and second terms is to the first, as the difference between the first and last to the sum of all the terms except the last.

Let A, B, C, D, E, be in continued proportion, then $A \sim B : A = A \sim E : A + B + C + D$.

Since (V. Def. 15) $A : B = B : C$, $B : C = C : D$, and $C : D = D : E$; therefore (V. 12)

$$A : B = A + B + C + D : B + C + D + E.$$

Hence (V. D), by conversion, $A : A \sim B = A + B + C + D : A \sim E$; for Prop. D of V. is true when the differences of A, B, and of C, D, are taken in the case of $B > A$, and consequently $D > C$; also the difference between the third and fourth terms above is evidently $A \sim E$. By inversion of the last analogy,

$$A \sim B : A = A \sim E : A + B + C + D.$$

EXERCISE III.—THEOREM.

If the same magnitude be added to the terms of a ratio, it will be unchanged, increased, or diminished, according as it is a ratio of equality, minority, or majority.

Let A, B, be the terms of a ratio, and C a third quantity; then,

First, if $A = B$, $A + C : B + C = A : B$;
Second, if $A < B$, $A + C : B + C > A : B$;
Third, if $A > B$, $A + C : B + C < A : B$.

First. Since $A = B$, therefore $A + C = B + C$; therefore (V. 7) $A + C : C = B + C : C$, and by alternation, $A + C : B + C = C : C$. It is similarly proved that $A : B = C : C$; and consequently (V. 11) $A + C : B + C = A : B$.

Second. Here $A < B$; and if $B - A = D$, then $B = A + D$.

Let p be such a number that $pC > A$; also, let nD be the least multiple of D that exceeds pA, so that
$pA < nD$; and hence $pA + A > nD$, or $= nD$;
and consequently $pA + pC > nD$, or $p(A + C) > nD$.

Let m be a number, such that $m = n + p$, or $p = m - n$;
then, since $pA < nD$, $p = m - n$, and $D = B - A$;
therefore $(m - n) A < n (B - A)$;
or, $mA - nA < nB - nA$;
whence $mA < nB$, by adding nA to each of the preceding unequals.

Again, since $p(A + C) > nD$,
therefore $pA + pC > n(B - A)$;
or, $mA - nA + mC - nC > nB - nA$;
that is, $mA + mC > nB + nC$, by adding $nA + nC$ to each of these unequals.
Wherefore $m(A + C) > n(B + C)$.

But it was proved above that $mA < nB$; consequently (V. Def. 14) $A + C : B + C > A : B$.

Third. Here $A > B$; and if $A - B = D$, then $A = B + D$.

Let p be such a number that $pC > B$; also, let mD be the least multiple of D that exceeds pB, so that

$pB < mD$; and hence $pB + B > mD$, or $= mD$;

and consequently $pB + pC > mD$, or $p(B + C) > mD$.

Let n be a number such that $n = m + p$, or $p = n - m$;

then, since $pB < mD$, $p = n - m$, and $D = A - B$;

therefore $(n - m)B < m(A - B)$;

or, $nB - mB < mA - mB$;

whence $nB < mA$, or $mA > nB$, by adding mB to each of the preceding unequals.

Again, since $p(B + C) > mD$,

therefore $pB + pC > m(A - B)$;

or, $nB - mB + nC - mC > mA - mB$;

that is, $nB + nC > mA + mC$, by adding $mB + mC$ to each of these unequals.

Wherefore $n(B + C) > m(A + C)$, or $m(A + C) <$

But it was proved above that $mA > nB$; consequently (V. 14)

EXERCISE IV.—THEOREM.

The difference of the successive terms of a series of continued proportionals are also in continued proportion.

Let $A : B = B : C$, $B : C = C : D$, and $C : D = D : E$; then, if $A > B$, $A - B : B - C = B - C : C - D$, and $B - C : C - D = C - D : D - E$.

For since $A : B = B : C$,

therefore, by division, $A - B : B = B - C : C$;

and alternately, $A - B : B - C = B : C$.

Again, because $B : C = C : D$, it is similarly proved that

$$B - C : C - D = C : D.$$

But by hypothesis, $B : C = C : D$; consequently, by equal ratios (V. 11), $A - B : B - C = B - C : C - D$.

It is similarly proved that $B - C : C - D = C - D : D - E$; consequently $A - B$, $B - C$, $C - D$, and $D - E$, are in continued proportion.

The same is similarly proved when $A < B$; that is, when the series of magnitudes are decreasing. In the above case, when $A > B$, the series is evidently increasing, for when $A > B$, then $B > C$, also $C > D$, and $D > E$.

EXERCISE V.—THEOREM.

The first term of an infinite decreasing series of quantities in continued proportion, is a mean proportional between its excess above the second and the sum of the series.

For whatever be the number of terms of the series, it is shown, as in the 2d Exercise, that, Z denoting the last term, S the sum of all the terms, and A, B, the first and second terms, $A - B : A = A - Z : S - Z$.
But the series may be so far extended, that the last term Z will be less than any assignable quantity, however small; hence the limits of the values of $A - Z$ and $S - Z$ are A and S; therefore

$$A - B : A = A : S.$$

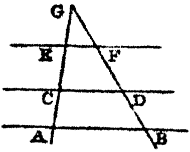
SIXTH BOOK.

EXERCISE I.—THEOREM.

Lines that intersect any three parallel lines are cut proportionally.

Let the two lines AE, BF, intersect the three parallel lines AB, CD, EF; then $AC : CE = BD : DF$.

For produce AE, BF, to meet in G; then (VI. 2) since CD is parallel to AB, a side of the triangle ABG,



$$AC : CG = BD : DG;$$

alternately, $AC : BD = CG : DG$.
Again, because CD is parallel to EF, a side of the triangle EFG, therefore

$$CE : CG = DF : DG;$$

and alternately, $CE : DF = CG : DG$.

But it was proved above that $AC : BD = CG : DG$;
wherefore, by equal ratios, $AC : BD = CE : DF$;
or alternately, $AC : CE = BD : DF$.

EXERCISE II.—THEOREM.

If a straight line be divided into three segments, such that the rectangle under the whole line and the middle segment is equal to that under the extreme segments, the line is cut harmonically.

Let the line MN be cut into three segments in P and Q, such that $MN \cdot PQ = MP \cdot QN$; then will $MN : NQ = MP : PQ$.

Since $MN \cdot PQ = MP \cdot QN$, therefore (VI. 16) the sides of these rectangles are reciprocally proportional; hence

$$MN : NQ = MP : PQ.$$

EXERCISE III.—THEOREM.

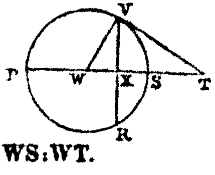
If from any point in the circumference of a circle a perpendicular be drawn on any radius, and a tangent from the same point to meet the radius produced, the radius will be a mean proportional between its segments, intercepted between the centre and the points of concourse.

Let PVS be a circle, V any point in its circumference, PS a diameter, VX a perpendicular on it from V, and VT a tangent meeting the diameter produced in T; then $WX : WS = WS : WT$.

For join V, W; then, since angle W is common to the triangles WVX, WTV, and angles WVT, WXV, are right angles, therefore (I. 32) their remaining angles are equal, and the triangles are equiangular; hence (VI. 4)

$$WX : WV = WV : WT;$$

and since $WV = WS$, $WX : WS = WS : WT$.



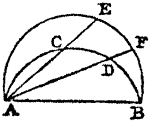
EXERCISE IV.—THEOREM.

If arcs of different circles have a common chord, lines diverging from one of its extremities will cut the arcs proportionally.

Let the arcs AEB, ACB, of two different circles have a common chord AB; then if the lines AC, AD, be drawn, BF : FE = BD : DC.

For (VI. 33) BF : FE = angle BAF : angle EAF; and also BD : DC = angle BAF : angle EAF; for angles BAF, EAF, are just the angles BAD, CAD.

Consequently, from equal ratios (V. 11),
BF : FE = BD : DC.

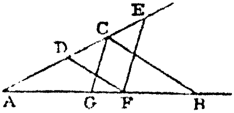


EXERCISE V.—PROBLEM.

To cut a straight line harmonically.

Let AB be the given line to be divided harmonically.

Draw any line AE, and on it take any two segments AD and DC, and make CE = CD; join CB, and draw DF parallel to CB; join EF, and draw CG parallel to EF, and AB is cut harmonically in G and F.



For since DF is parallel to CB, the triangles ABC, AFD, are similar; and for a similar reason, AEF and ACG are also similar; consequently

AB : BF = AC : CD (VI. 4);
but AC : CD = AC : CE, because CE = CD;
also AC : CE = AG : GF (VI. 4);

since the second ratio of the first of these three analogies, and the first ratio of the third, are equal, the first ratio must be equal to the last; that is,

AB : BF = AG : GF;

whence AB is cut harmonically.

Or, more concisely thus:

AB : BF = AC : CD (VI. 4) = AC : CE = AG : GF (VI. 4); hence AB : BF = AG : GF, or AB : AG = BF : GF (V. 16).

SCHOLIUM.—It is evident that the line AB could be cut harmonically in any given ratio; for if the ratio of any two lines M and N is the given ratio, AC and CD could be made respectively equal to M and N, and then AB would be cut harmonically in the given ratio.

A line can therefore be cut internally and externally in the same ratio (II. Def. 3, 4).

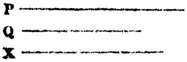
For if any line AF were given, it could be cut in G, and produced to B, so that $AB : BF = AG : GF$, by joining DF, and drawing CB parallel to DF, and then joining EF, and drawing CG parallel to EF. (See VI. A, Scholium 1).

EXERCISE VI.—PROBLEM.

To find a line such, that the first of two given lines shall be to the second as the square of the first to the square of the required line.

Let P, Q, be two given lines; it is required to find a third line X, such that $P : Q = P^2 : X^2$.

Find a line X, a mean proportional between P and Q (VI. 13), then $P : X = X : Q$; and consequently $P : Q$ in the duplicate ratio of P to X (V. Def. 18); but this latter ratio is that of P^2 to X^2 (VI. 20, Cor. 3); hence $P : Q = P^2 : X^2$.

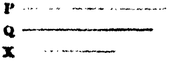


EXERCISE VII.—PROBLEM.

To find a line such, that the first of two given lines shall be to it as the square of the first to the square of the second.

Let P, Q, be two given lines; to find a third line X, such that $P : X = P^2 : Q^2$.

Find a third proportional X, to the lines P and Q; then $P : Q = Q : X$; and therefore, as in the preceding exercise (V. Def. 18, and VI. 20, Cor. 3), $P : X = P^2 : Q^2$.

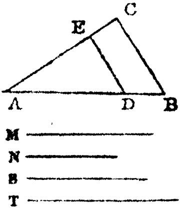


EXERCISE VIII.—PROBLEM.

From a given angle, to cut off a triangle equal to a given space, so that its sides about that angle shall have a given ratio.

Let BAC be a given angle, M, N, two lines in a given ratio, and S a line whose square is the given space; to cut off, by a straight line DE, a triangle ADE $= S^2$, and such that AD : AE = M : N.

Make AB = M, and AC = N, and join BC; then AB : AC is the given ratio. Find the side T of a square equal to the triangle ABC (II. 14); and find AD a fourth proportional to T, S, and AB (VI. 12), and draw DE parallel to BC; then ADE is the required triangle.



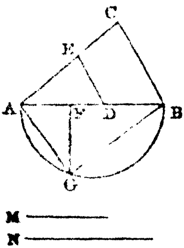
For (VI. 22) $T^2 : S^2 = AB^2 : AD^2$; and (VI. 22, Cor. 3) triangle ABC : triangle ADE $= AB^2 : AD^2$; hence (V. 11) triangle ABC : triangle ADE $= T^2 : S^2$. But triangle ABC $= T^2$; hence triangle ADE $= S^2$; also AD : AE = AB : AC, since DE is parallel to BC (VI. 2); wherefore AD : AE = M : N.

EXERCISE IX.—PROBLEM.

To cut off from a given triangle another similar to it, and in a given ratio to it.

Let ABC be a triangle, and M, N, two lines, to cut off from ABC another triangle ADE, similar to ABC, so that ADE : ABC = M : N.

On AB describe the semicircle ABG; and cut off AF, so that AB : AF = N : M (VI. 12); draw FG perpendicular to AB, and join AG and BG; make AD = AG; draw DE parallel to BC, and ADE is the required triangle.

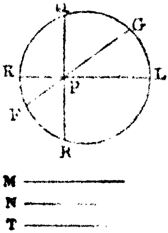


For (VI. 8, Cor.) AB : AF $= AB^2 : AG^2$; but AB : AF = N : M, and AD = AG; therefore N : M $= AB^2 : AD^2$. Again, the triangle ABC : triangle ADE $= AB^2 : AD^2$ (VI. 20, Cor. 3); wherefore triangle ABC : triangle ADE = N : M.

EXERCISE X.—PROBLEM.

Through a given point in a circle, to draw a chord whose segments shall have a given ratio.

Let KFL be a given circle, and the ratio of the lines M, N, the given ratio, and P the given point; to draw a chord GF through P, so that GP : PF = M : N.



Through P draw the diameter KL, and PQ perpendicular to it; make AB (fig. to 9th Ex.) = M, and AC = N, and find a line T a mean proportional to AB and AC; then find AD a fourth proportional to T, PQ, and AB; and draw DE parallel to BC; then AD, AE, are the required segments of FG. Make PG = AD, and produce PG to F, and PF is the required chord.

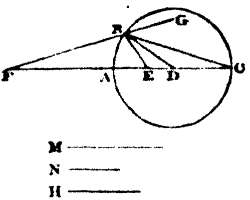
Since $T^2 : PQ^2 = AB^2 : AD^2$ (VI. 22); and since the rectangles BA · AC and DA · AE are similar, for BA : AC = AD : AE, therefore $BA \cdot AC : DA \cdot AE = AB^2 : AD^2$ (VI. 22, Cor. 3); consequently $BA \cdot AC : DA \cdot AE = T^2 : PQ^2$; but, by construction, BA : T = T : AC; wherefore (VI. 16) $BA \cdot AC = T^2$; therefore (V. 14) $DA \cdot AE = PQ^2$; but $PG \cdot PF = PQ^2$ (III. 35); also AD = PG; wherefore PF = AE. Again, by construction, AD : AE = M : N; therefore PG : PF = M : N.

EXERCISE XI.—PROBLEM.

Given the base, the altitude, and the ratio of the sides of a triangle; to construct it.

Let FE be the base of a triangle, the altitude a line H, and the ratio of the sides that of two lines M and N; to construct the triangle.

Place M and N contiguous, and in a line, then cut FE in A in the ratio of M to N (VI. 10); produce FE, and cut it



in C, so that $FC : CE = M : N$ (Ex. 5, Scholium). Then on AC, as a diameter, describe the circle ABC; draw from A a perpendicular to FA, and equal H (not represented in the figure), and through its upper extremity draw a parallel to FE, cutting the circle ABC in B (as in the 17th Ex. of the Fourth Book); then draw **BF**, BE, and **BEF** is the required triangle.

Join AB, BC, and draw the radius BD; then, since by construction $FA : AE = M : N$, and $FC : CE = M : N$; therefore $FC : CE = FA : AE$; hence,

by alternation (V. 16), $CF : FA = CE : EA$;

by mixing (V. G), $CF + FA : CF - FA = CE + EA : CE - EA$.

But

$$CF + FA = 2\,FA + 2\,AD = 2\,FD,$$
$$CF - FA = AC = 2\,AD,$$
$$CE + EA = CA = 2\,AD,$$
$$CE - EA = AD + DE - EA = AE + 2\,DE - EA = 2DE;$$

wherefore $2\,FD : 2\,AD = 2\,AD : 2\,DE$,

or (V. 15) $FD : AD = AD : DE$, or $ED \cdot DF = AD^2$; it therefore follows, by VI. F, that $FB : BE = FA : AE$; and consequently FBE is the required triangle.

Cor. 1.—The circle ABC is the locus of the vertices of all the triangles that can be constructed on FE as a base, and having their sides in the given ratio of M to N.

Cor. 2.—If a line, as FC, be cut in A and E, so that $FC : CE = FA : AE$; then also $FC : FA = CE : EA$.

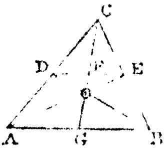
Cor. 3.—If the segment AC, made up of the middle and an extreme segment, be bisected in D, then DF, DA, and DE, are in continued proportion; and conversely.

EXERCISE XII.—THEOREM.

If from the extremities of the base of a triangle, lines be drawn bisecting the opposite sides, they will divide each other in the ratio of two to one.

Let ABC be a triangle, and AE, BD, lines from A and B, bisecting BC, CA, in E and D; then $AO : OE = 2 : 1$, and $BO : OD = 2 : 1$.

For join DE; then since $CD = DA$, and $CE = EB$, therefore $CD : DA = CE : EB$, and consequently (VI. 2) DE is parallel to AB; wherefore angle $CDE = CAB$, and $CED = CBA$ (I. 29), and the triangles CDE, CAB, are equiangular; therefore (VI. 4) $CD : DE = CA : AB$;
or alternately, $CD : CA = DE : AB$;
but $CD = \frac{1}{2} CA$; therefore $DE = \frac{1}{2} AB$ (V. C).
Now, triangles AOB, DOE, have the vertical angles at O equal (I. 15), and the alternate angles at D and B are equal, wherefore the triangles are similar; hence (VI. 4) $AO : OE = AB : DE$; but AB = twice DE, therefore $AO = 2\,OE$; and it is similarly proved that $BO = 2\,OD$.



EXERCISE XIII.—THEOREM.

If a line be drawn parallel to the base of a triangle to meet the sides, and the alternate extremities of this line and of the base be joined, the line drawn from the vertex through the intersection of the connecting lines will bisect the base, and will be cut harmonically.

Let ABC (fig. to last Ex.) be a triangle, and DE any line parallel to the base AB, and cutting the sides in D and E, then CO produced, bisects the base in G, and CG is cut harmonically; or $CG : CF = GO : OF$.

The triangles CDF and CAG are similar, since DE is parallel to AB; hence

$$CG : CF = CA : CD \text{ (VI. 4, and V. 16).}$$

Again, as in the last Exercise, it is proved that the triangles ABC, DEC, are similar; hence

$$CA : CD = AB : DE;$$

but, as was proved in the preceding Exercise, the triangles ABO, DEO, are similar; therefore

$$AB : DE = AO : OE.$$

Now, the triangles AOG, FOE, are similar, since their vertical angles at O are equal, and the alternate angles at A and E; wherefore $AO : OE = GO : OF$.

In the last four analogies, the second ratio in each is the same as the first of the succeeding analogy; hence the first ratio is equal to the last; or

$$CG : CF = GO : OF;$$

that is, CG is cut in harmonic proportion.

Again, in the similar triangles CGB, CFE,

$$GB : FE = CG : CF;$$

and in the similar triangles AOG, FOE,

$$AG : FE = GO : OF;$$

but it was already proved that $CG : CF = GO : OF$; wherefore, by equal ratios (V. 11),

$$GB : FE = AG : FE; \text{ hence (V. 14) } AG = GB.$$

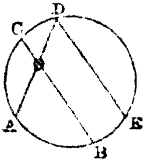
EXERCISE XIV.—THEOREM.

The inclination of two chords of a circle is measured by half the sum, or half the difference, of the intercepted arcs, according as they intersect internally or externally.

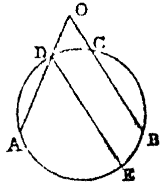
Let ABED be a circle, and AD, BC, two chords intersecting in O; then,

First, When the section is internal, angle AOB is measured by half the sum of the intercepted arcs AB, CD.

For draw DE parallel to BC; then (III. 9th Ex.) arc $CD = BE$; wherefore arc $AE = AB + BE = AB + CD$. But angle $AOB = D$, and D is equal half the angle at the centre, standing on the arc AE (III. 20), which may be measured by that arc (VI. 33, Cor. 2); hence angle D or AOB is measured by half the arc AE, or by half the sum of AB and CD.



Secondly, Let the section be external, then angle AOB is measured by half the difference of the intercepted arcs AB, CD.



For if DE be drawn parallel to BC, then, as in the preceding case, arc CD = BE, and angle O = ADE. But angle ADE is measured by half the arc AE, which is the difference between the arcs AB and BE, or AB and CD; wherefore angle O is measured by half the difference between the arcs AB and CD.

EXERCISE XV.—THEOREM.

If three lines be in continued proportion, the first is to the third as the square of the difference between the first and second, to the square of the difference between the second and third.

Let A, B, C, denote three lines, such that $A : B = B : C$. then $A : C = (A \sim B)^2 : (B \sim C)^2$.

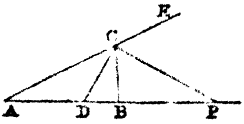
For by conversion (V. D.), $A : A \sim B = B : B \sim C$;
by alternation (V. 16), $A : B = A \sim B : B \sim C$;
consequently (VI. 22, Cor.), $A^2 : B^2 = (A \sim B)^2 : (B \sim C)^2$.
But (V. Def. 18, and VI. 20, Cor. 3), $A : C = A^2 : B^2$;
therefore, by equal ratios (V. 11), $A : C = (A \sim B)^2 : (B \sim C)^2$.

EXERCISE XVI.—THEOREM.

If a line bisect the angle adjacent to the vertical angle of a triangle, and meet the base produced, the difference between the square of that line and the rectangle under the external segments of the base, is equal to the rectangle under the sides of the triangle.

Let ABC be a triangle, CP a line bisecting angle BCE adjacent to the vertical angle, and cutting the base AB produced in P, then $AP \cdot PB - CP^2 = AC \cdot CB$.

Draw CD, bisecting angle ACB; then since angles ACB, BCE, together are equal to two right angles, their halves DCB, BCP, together will be equal to one right angle, or DCP = a right angle; hence $CD^2 + CP^2 = DP^2$. But (VI. B), $AC \cdot CB = AD \cdot DB + CD^2$;



adding CP^2 to these equals,

$$AC \cdot CB + CP^2 = AD \cdot DB + CD^2 + CP^2\;;$$

$$= AD \cdot DB + DP^2\text{; since it was}$$

above shown that $CD^2 + CP^2 = DP^2$.

Now (II. 4), $DP^2 = DB^2 + BP^2 + 2\,DB \cdot BP$,

$$= DB^2 + BP^2 + DB \cdot BP + DB \cdot BP.$$

Hence $AD \cdot DB + DP^2 = AD \cdot DB + DB^2 + DB \cdot BP + DB \cdot BP + BP^2$.

But (II. 1) $AP \cdot DB = (AD + DB + BP)\,DB = AD \cdot DB + DB^2 + BP \cdot DB$;

and (II. 3) $DP \cdot BP = DB \cdot BP + BP^2$;

consequently $AD \cdot DB + DP^2 = AP \cdot DB + DP \cdot BP$.

Now (VI. A., Cor.) AP is cut harmonically in D and B,

hence $AP : PB = AD : DB$;

therefore (VI. 16) $AP \cdot DB = AD \cdot BP$.

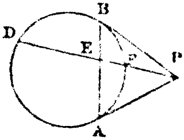
Wherefore $AD \cdot DB + DP^2 = AD \cdot BP + DP \cdot BP = (AD + DP)\,BP = AP \cdot BP$.

And since it was formerly proved that $AD \cdot DB + DP^2 = AC \cdot CB + CP^2$; consequently $AC \cdot CB + CP^2 = AP \cdot BP$; or the difference between $AP \cdot BP$ and CP^2, is equal to $AC \cdot CB$.

EXERCISE XVII.—THEOREM.

If two tangents and a secant be drawn to a circle from a point without it, and the points of contact be joined by a straight line, the secant will be cut harmonically.

Let ABD be a circle, PB, PA, two tangents to it from any point P; join AB, and draw any secant PD from P, and it is cut harmonically; that is, $DP : PF = DE : EF$.



For the tangents AP, BP, are equal (III. 37, Cor.); hence ABP is an isosceles triangle; therefore (II. D.) $PB^2 - PE^2 = BE \cdot EA$, and (III. 35) $BE \cdot EA = DE \cdot EF$; consequently $PB^2 - PE^2 = DE \cdot EF$; that is, PB^2 exceeds PE^2 by the rectangle $DE \cdot EF$; or

$$PB^2 = PE^2 + DE \cdot EF.$$

But (III. 36) $PB^2 = PD \cdot PF$;

consequently $PD \cdot PF = PE^2 + DE \cdot EF$.

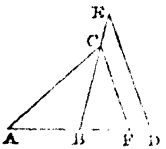
Now (II. 1) $PD \cdot PF = DE \cdot PF + EF \cdot PF + PF^2$;
and (II. 4) $PE^2 = EF^2 + PF^2 + 2\,EF \cdot PF$;
therefore $DE \cdot PF + EF \cdot PF + PF^2 = EF^2 + PF^2 + 2\,EF \cdot PF + DE \cdot EF$.
From both these equal quantities take away the equals; namely, $EF \cdot PF$, and also PF^2, and there remains $DE \cdot PF = EF^2 + EF \cdot PF + DE \cdot EF$.
But (II. 1) $EF \cdot DP = EF \cdot DE + EF \cdot EF + EF \cdot PF$;
consequently $DE \cdot PF = DP \cdot EF$;
or (VI. 16) $DP : PF = DE : EF$;
and therefore (VI. Def. 5) the secant PD is harmonically divided.

EXERCISE XVIII.—THEOREM.

If two triangles have two angles together equal to two right angles, and other two angles equal, the sides about their remaining angles are proportional.

Let ABC, BDE, be two triangles having the angles at B supplementary, and the vertical angles ACB, BED, equal, then shall

$$CA : AB = ED : DB.$$



For having placed the triangles with their supplementary angles contiguous, so that the sides opposite to the equal angles may be in the same straight line ABF, then draw CF parallel to ED, and therefore (I. 29) angle $BCF = BED = ACB$; consequently angle ACF is bisected by BC; and hence (VI. 3)

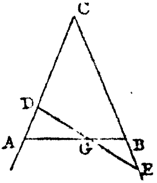
$$AC : CF = AB : BF;$$

or alternately (V. 16), $AC : AB = CF : BF$.
But triangles BCF, BED, are evidently similar, and hence $CF : BF = ED : DB$; whence, by equal ratios (V. 11), $AC : AB = ED : DB$.

EXERCISE XIX.—THEOREM.

If a line be drawn through any point in the base of an isosceles triangle, so as to cut off from one side, and add to the other, equal segments, it will be bisected by the base.

Let ABC be an isosceles triangle, and let the line DGE cut off the segment AD from one side AC, and add the equal segment BE to the other side BC, then will DE be bisected in G.

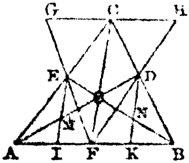


For angle A = ABC (I. 5); but ABC + ABE = two right angles; hence A + GBE = two right angles; also the vertical angles at G are equal; it follows, therefore (Ex. 18), that in the two triangles ADG, BGE, the sides about the angles at D and E are proportional, or AD : DG = BE : EG; whence (V. 14) DG = GE, because AD is given = BE.

EXERCISE XX.—THEOREM.

If from the angular points of a triangle, lines be drawn through any point within it to meet the opposite sides, and if from the point of section of the base, lines be drawn through the other two points of section to meet a line drawn through the vertex parallel to the base, the intercepted portion of the latter is bisected in the vertex.

Let ABC be the triangle, and O the point within it; AD, BE, CF, the lines through it from the angular points, GH a line parallel to AB; and produce FD, FE, to G and H; then is GC = CH.



For draw EI, DK, parallel to CF, then the triangles AME, AOC, are similar; as also AIM and AFO; BOC and BND; and also BOF and BNK; hence (VI. 4)

EM : CO = AM : AO, and AM : AO = MI : OF;

hence, from equal ratios, EM : CO = MI : OF;

and alternately, EM : MI = CO : OF.

In the same manner it is proved that DN : NK = CO : OF;

whence, from equal ratios, EM : MI = DN : NK;

by addition, EM : EI = DN : DK;

or by alternation, EI : DK = EM : DN.

But because the triangles EOM, DON, are manifestly similar, EM : DN = OM : OD,

and (VI. 2) OM : OD = IF : FK ;
whence, from equal ratios, EI : DK = IF : FK,
and alternately, EI : IF = DK : FK.
But triangles EIF and GCF are evidently similar, since EI is parallel to CF, and CG to IF; and hence angle IEF = CFE, and EFI = FGC; and it is similarly shown that triangles DKF, CFH, are similar;
whence EI : IF = FC : CG,
and DK : KF = FC : CH ;
whence, from equal ratios, FC : CG = FC : CH ;
consequently (V. 14) CG = CH.

The proportions may be stated more concisely thus, after proving the various triangles similar as above:
EM : MI = CO : OF = DN : NK, or EM : MI = DN : NK.
By addition and inversion, EI : EM = DK : DN.
Or EI : DK = EM : DN = EO : ON = IF : FK ;
whence EI : IF = DK : FK ;
and FC : CG = EI : IF = DK : KF = FC : CH ;
and hence CG = CH.

EXERCISE XXI.—THEOREM.

If from the extremities of the base of a triangle, lines be drawn through any point in the perpendicular to meet the sides, lines joining the points of section of the sides with that of the base, will make equal angles with the base.

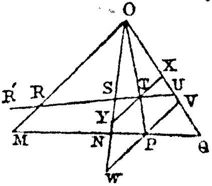
In the preceding Exercise, it was proved that CG = CH; and hence, when CF is perpendicular to AB, or to its parallel GH, then the two triangles FCG, FCH, have the sides FC, CG, respectively equal to FC, CH; and the contained angles being then right angles, therefore the triangles are everyway equal; and hence angle CFG = CFH.

EXERCISE XXII.—THEOREM.

Harmonicals cut all straight lines that intersect them harmonically.

Let OM, ON, OP, OQ, be harmonicals, and RU any line intersecting them, then RU is cut harmonically.

For the harmonicals must cut some line harmonically (VI. Def. 8); let MQ be this line, and draw VW and XY parallel to OM. Then, since VW is parallel to OM, the exterior angle P and interior M are equal, and also the angle Q is common to the two triangles PQV and MQO; they are therefore similar (I. 32, and VI. 4). Again, in the triangles PNW and MNO, the alternate angles at P and M are equal, and the vertical angles at N, therefore these two triangles are similar; and for exactly similar reasons the triangles TUX, RUO, STY, and RSO, are similar;



wherefore $MO : PV = MQ : PQ$ (VI. 4),
and $MQ : PQ = MN : NP$ (by hypothesis);
also $MN : NP = MO : PW$ (VI. 4);
and since the second ratios of the first and second of these three analogies are the same as the first of the second and third respectively, the first ratio must be equal to the last;
that is, $MO : PV = MO : PW$;
and consequently (V. 14) $PV = PW$. … [2]

Again, since XY is parallel to VW, the triangles OTY, OPW, are similar, and also OTX, OPV;
consequently $PW : TY = OP : OT$ (VI. 4),
and $OP : OT = PV : TX$;
whence, by equal ratios, $PW : TY = PV : TX$ (V. 11);
but $PW = PV$ by [2] above; hence $TY = TX$ (V. 11).

By similar triangles, $RU : TU = RO : TX$ (VI. 4),
and $RO : TX = RO : TY$, for $TX = TY$;
also $RO : TY = RS : TS$ (VI. 4);
hence, by equal ratios, $RU : TU = RS : TS$;
and therefore RU is cut harmonically.

The demonstration may be more concisely stated thus, after proving the various triangles concerned to be similar:

$MO : PV = MQ : PQ = MN : NP = MO : PW$; therefore $PV = PW$.

Again, $TY : PW = OT : OP = TX : PV$; but $PW = PV$; hence $TY = TX$;

also $RU : TU = RO : TX = RO : TY = RS : TS$; or RU is cut harmonically.

COR.—If the harmonicals OM, ON, OP, OQ, cut a line MQ harmonically, and if another line R′U be cut harmonically, so that one of the harmonicals passes through one of its extremities U, and other two of them through two of the points of harmonic section S and T, then the fourth harmonical OM must pass through its other extremity R.

For if not, let R′ be the other extremity of the line that is cut harmonically, then

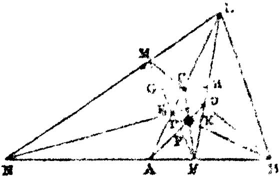
$R'U : TU = R'S : ST$, by hypothesis;
but $RU : TU = RS : ST$, by the Exercise;
whence $R'U : R'S = RU : RS$ (V. 22);
and $R'U : SU = RU : SU$, by conversion;
hence $R'U = RU$ (V. 14);

wherefore R and R′ must coincide; that is, the point R in the line OM must be the other extremity of the line.

EXERCISE XXIII.—THEOREM.

If lines be drawn from the angular points of a triangle, through any point within it, to meet the opposite sides, and the points of section be joined, the former lines will be cut harmonically; and if these lines be produced to meet the sides produced, the latter will be cut harmonically.

Let ABC be the triangle, and O the point within it, and through O draw AD, BE, CF, and join DE, EF, and FD; then AD, BE, CF, are cut harmonically.



For produce FE and FD to G and H, and draw IK and GCH parallel to AB. Then, since GH is parallel to IK, the triangles GCF, IOF, are similar, and also the triangles HCF and OKF; therefore

$$GC : IO = CF : OF, \text{ and } CH : OK = CF : OF;$$

hence (V. 11) $GC : IO = CH : OK$, and $GC = CH$ (Ex. 20); hence (V. 14) $IO = OK$.

Again, the triangles ADF, ODK, are similar, and also APF and OPI, because IK is parallel to AF;
wherefore AD : DO = AF : OK, and AP : PO = AF : IO;
but the second ratios of these last two analogies are equal, because OK = IO;
whence (V. 11) AD : DO = AP : PO;
consequently AD is cut harmonically; and in the same manner it can be shown that BE and CF are similarly cut.

Let DE, EF, FD, be produced to meet the opposite sides in N, M, and L, and these produced sides AL, BM, BN, are cut harmonically.

For the triangles ALF, CLH, are similar, and so are AEF, CEG, since GH is parallel to AF;
whence AL : LC = AF : CH, and AE : EC = AF : CG;
but the second ratios are equal, since CG = CH,
whence (V. 11) AL : LC = AE : EC;
consequently AL is cut harmonically; and in the same manner it is proved that BM and BN are cut harmonically.

EXERCISE XXIV.—THEOREM.

If through any point within a triangle lines be drawn from the angular points to meet the opposite sides, and the lines joining the points of section be produced to meet the sides produced, the three points of concourse are in one line.

Let ABC (fig. to last Ex.) be the given triangle, O the given point, and L, M, N, the points of concourse; they are in one straight line.

For join LB; then, because the lines LB, LF, LA, and LN, cut BN harmonically (last Ex.), they are harmonicals; and because three of the harmonicals cut the line BM in three points, B, D, C, of harmonic section, therefore the fourth harmonic, LN, must pass through M, the extremity of BM (22d Ex., Cor.); consequently the points L, M, N, are in one straight line.*

---

* The only demonstration that I have seen of this theorem is incorrect, as it assumes DLN to be a triangle, which implicitly assumes LMN to be a straight line.

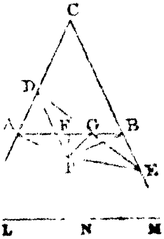
Scholium.—The propositions stated in the enunciations of this and the preceding exercise are true, when the point is taken without the triangle.

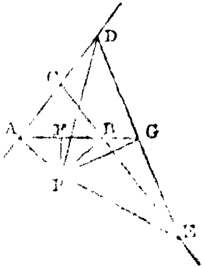
EXERCISE XXV.—PROBLEM.

To draw a straight line so that the part of it intercepted between one side of a given isosceles triangle and the other side produced, shall be equal to a given line, and be bisected by the base.

Let ABC be the given isosceles triangle (first figure), and LM the given line; to draw a line DE = LM, so that it may be bisected by AB.

Draw AP, BP, perpendiculars to AC, BC, and PF perpendicular to AB, bisect LM in N, and find a line GP a fourth proportional to AF, LN, and PF, and from P, with radius PG, cut AB in G; join PG, and draw DGE perpendicular to PG, and DE is the required line.





For angles PAD, PGD, are right angles, and hence PD is the diameter of a circle passing through the points ADGP; consequently angles PAG, PDG, are angles in the same segment of this circle, and therefore equal (III. 21); and consequently the triangles PAF, PDG, are similar, and therefore PF : FA = PG : GD; but PF : FA = PG : LN, by construction; consequently GD = LN.

Now in the triangles PAF, PBF, the angles at F are equal, being right angles, and those at A and B are equal, being the complements of the equal angles CAB, CBA, and the side PF is common to both; hence the triangles are everyway equal (I. 26); therefore AF = FB.

Again, angles PGE, PBE, being right angles, are equal; hence PE is the diameter of a circle passing through the points P, G, B, and E; and hence angles PBG, PEG, would be in the same segment of this circle, and are consequently equal (III. 21); hence the two triangles PGE, PFB, are similar;
and therefore PF : FB = PG : GE; but PF : FB = PG : LN by construction, since AF = FB; therefore DG = LN = GE.

The same proof applies exactly to the second case, represented by the second figure; that is, when PG is greater than PB or PA, and the line DGE must be drawn without the triangle; only in this case angle PEG + PBG = two right angles (III. 22), and PBF + PBG = two right angles; wherefore PBF = PEG.

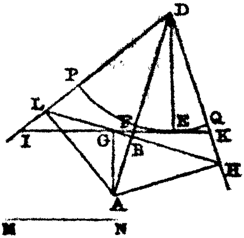
Cor.—Any line PG being drawn from P to any point G in AB, and a line DE drawn perpendicular to it, the latter will be bisected.

EXERCISE XXVI.—PROBLEM.

Given the altitude, the vertical angle, and the sum or difference of the sides of a triangle; to construct it.

Let LDH (first figure) be the vertical angle, LD = DH = half the sum of the sides, and DP = P, the altitude of the triangle.

From centre D, with radius DP, describe the arc PEQ; draw LA, HA, perpendicular to DL and DH; join LH and AD. Find MN a third proportional to AB and DP; that is,
let AB : DP = DP : MN, or
$AB \cdot MN = DP^2$ ... [1]
and then find the point F in AD, so that DF may be a third proportional to AF and MN (II. Ex. 14); that is,



let AF : DF = DF : MN, or $AF \cdot MN = DF^2$ ... [2]

next find a line AG a mean proportional between AB and AF; that is, so that

$AB : AG = AG : AF$, or $AB \cdot AF = AG^2$ ... [3]

then draw AG from A, so as to terminate in the line LH; join GF, and produce GF both ways to I and K, and IDK will be the required triangle.

For the triangles ADL, ADH, have the side $DL = DH$, and AD common, and the angles at L and H right angles; hence (I. C) they are everyway equal; therefore angle $DAL = DAH$. Again, the triangles ABL, ABH, have the side $AL = AH$, and AB common, and the angle $BAL = BAH$; therefore these triangles are everyway equal (I. 4); consequently the angles at B are right angles. Now by [3] the triangles ABG, AGF, must be similar, as the sides about their common angle BAG are proportional (VI. 6); hence angle $AGF = ABG =$ a right angle. Hence, if DE be a perpendicular on IK, the triangles AGF, DEF, are similar, the vertical angles at F being equal, and the right angles at G and E;

therefore $AF : AG = DF : DE$ ... [4]

But $AF^2 : AG^2 = AF : AB$ (VI. 8, and 20, Cor. 2 and 3); or $AF^2 : AG^2 = AF \cdot MN : AB \cdot MN$, considering AF, AB, as the bases, and MN as the altitude of these two rectangles (VI. 1);

hence $AF^2 : AG^2 = DF^2 : DP^2$ by [2] and [1] above;

or $AF : AG = DF : DP$ (VI. 22, Cor.)

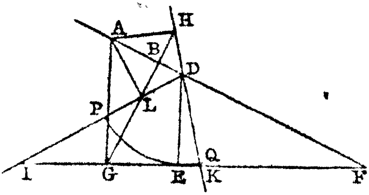
But $AF : AG = DF : DE$ by (4) above;

hence $DF : DP = DF : DE$, and $DE = DP$ (V. 14).

Consequently DE is a radius of DEQ; and being perpendicular to IK, the triangle DIK has the given altitude $= DP$.

Again, since IK is perpendicular to AG, therefore it is bisected, and consequently $IL = KH$ (Ex. 25, Cor.); therefore $ID + DK = DL + IL + DH - KH = DL + DH =$ sum of sides; and IDK is the given vertical angle; consequently IDH is the required triangle.

The second case is, when the difference of the sides are given instead of the sum. In this case, IDK in the second figure is the given vertical angle, as in the first case; but the isosceles triangle LDH has for its vertical angle the angle LDH supplementary to IDK, and its sides DL, DH, are each half the difference of the sides. AG is found as in the first case, the point F here being the intersection of AD and IK, as in the first case; and it can be proved here also (see 2d case of 25th Ex. and Cor.) that $IG = GK$, and $IL = KH = DK + DH$; whence



$$DI - DL = DK + DH, \text{ or } DI = DK + DL + DH;$$

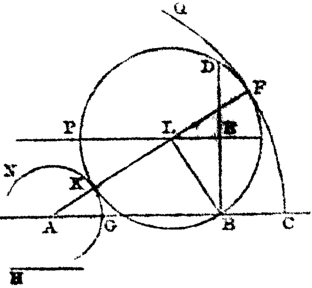
whence DI exceeds DK by $DL + DH$, or by the difference of the sides. The rest of the proof is the same as in the first case. Hence the triangle IDK has the given vertical angle, the given altitude $DE = DP$, and the given difference of sides; it is therefore the required triangle.

EXERÇISE XXVII.—PROBLEM.

Given the base, the altitude, and the sum or difference of the sides of a triangle; to construct it.

Let AB be the base, H the altitude, and AC the sum of the sides of a triangle; to construct it.

From A as a centre, with the radius AC, describe an arc CFQ of a circle; draw BD perpendicular to AB, and make BE, ED, each $= H$; find L, the centre of a circle BDF touching the arc CF in F, and passing through the points B, D (III. Ex. 14); join its centre L with A and B, and ABL is the required triangle.



For if PE be drawn perpendicular to BD, it will be a chord of the circle DFB (III. 3, Cor.), and therefore the centre L of this circle will lie in PE; and AL produced must pass through the point of contact F (III. 11); consequently ALF is a straight line, and = AC; but AL + LB = AL + LF; hence AL + LB = AC. Again, since PE and AB are both perpendicular to BD, they are parallel (I. 29); hence (I. 34, Cor. 1) the altitude of the triangle is = BE = H. ABL is therefore the required triangle.

When the difference of the sides is given instead of the sum, the construction is similar.

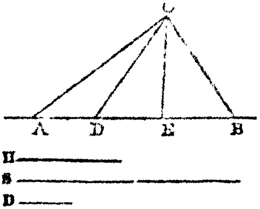
Let AG be the difference of the sides, and from the centre A, with the radius AG, describe the arc GKN; draw BD and PE, as in the preceding case, and describe a circle PBD through the points B, D, and touching the arc GKN in K; and its centre L will be the vertex of the triangle, and ABL the required triangle.

For the line AL, joining the centres, will pass through K; hence AK = AL — LK = AL — LB, or AL — LB = AG. Hence ABL has the given base, the given height, and the difference of its sides equal to the given difference; it is therefore the required triangle.

EXERCISE XXVIII.—PROBLEM.

The altitude, the difference of the angles at the base, and either the sum or difference of the sides of a triangle being given; to construct it.

Let H be the altitude of the triangle, V the difference of the angles at the base, and S the sum of the sides; to construct the triangle.



Construct, by the 26th Exercise, the triangle ACD having a vertical angle ACD = V, an altitude CE = H, and the sum of its sides AC, CD = S; produce AE to B, till EB = ED, and join BC; then ABC is the required triangle.

For the triangles CED, CEB, have the sides CE, ED, respectively equal to CE, EB, and the angles at E are equal; hence (I. 4) the triangles are everyway equal, and therefore CD = CB, and angle B = CDE. Now angle CDE = A + ACD; therefore ACD is the difference between CDE and A, or between B and A; that is, it is = to V. Also AC + CB = AC + CD = S. The triangle ABC, therefore, has the given altitude, the sum of its sides equal to a given line, and the difference between the angles at its base equal to a given angle; it is therefore the required triangle.

When the difference D between the sides is given instead of their sum; construct the triangle ACD so as to satisfy the conditions by the 26th Exercise, then construct the triangle ABC as in the preceding case; then since AC — CD = D, and AC — CD = AC — CB, therefore AC — CB = D; and consequently ABC is the required triangle.

EXERCISE XXIX.—PROBLEM.

The altitude, the difference of the segments of the base, and either the sum or difference of the two sides of a triangle, are given; to construct it.

Let H be the altitude, D the difference of the segments of the base, and S the sum of the sides of a triangle; to construct it.

Construct, by the 27th Exercise, the triangle ACD (fig. to last Ex.), having a base AD = D, an altitude CE = H, and the sum of its sides AC, CD = S; then complete the triangle ABC, as in the last Exercise, and it is the required triangle.

For it was proved, in the preceding Exercise, that CB = CD, and hence AC + CB = AC + CD; but AC + CD = S; therefore AC + CB = S; also AD = AE — ED = AE — EB. Hence the triangle ABC has the given altitude, the difference of the segments of its base equal to a given line D, and the sum of its sides equal to S.

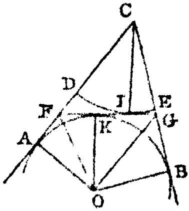
When the difference of the sides D is given instead of their sum, construct the triangle ACD by the same Exercise, namely, the 27th (second case), and complete the triangle ABC; and it can be shown, as in the first case above, that ABC is the required triangle.

EXERCISE XXX.—PROBLEM.

Given the altitude, the vertical angle, and the perimeter of a triangle; to construct it.

Let ACB be the given vertical angle, AC, CB, each equal to the semi-perimeter, and CD the altitude; to construct the triangle.

Draw OA, OB, from A and B perpendicular respectively to AC and BC; from C as a centre, with the radius CD, describe the arc DIE, and from O, with the radius OA, describe the arc AKB; draw FKIG a common tangent to these two arcs (IV. Exercise 7), K and I being the points of contact; then FCG is the required triangle.



For join OF, OG, OK, and CI; then the two triangles OAF, OKF, have OA = OK, and OF common, and the right angles A and K; hence (I. C, Cor.) the triangles are everyway equal, and therefore OF bisects the angles AFK; and it is similarly proved that OG bisects the angle BGK; wherefore (VI. K) AF = KF, and BG = KG; therefore CF + CG + FG = CF + FK + CG + KG = CF + FA + CG + GB = CA + CB = the given perimeter; also CI is perpendicular to FG (III. 18), and it is = CD. The triangle FCG, therefore, has the given altitude, the given vertical angle, and the given perimeter; it is therefore the required triangle.

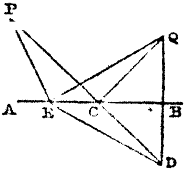
GEOMETRICAL MAXIMA AND MINIMA.

EXERCISE I.—PROBLEM.

A straight line and two points without it being given, to find a point in it such that the sum of the lines drawn from it to the given points shall be a minimum.

Let P, Q, be the given points, and AB the given line; to find a point, as C, in AB such that PC + CQ shall be the least possible.

Draw BQ perpendicular to AB, and produce QB till BD = BQ; join DP, and C is the required point.



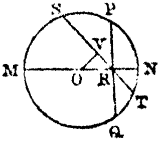
For join CQ; then in the two triangles BCQ, BCD, the side BD = BQ, and BC is common to both, and the angles at B are equal, being right; consequently (I. 4) the triangles are everyway equal; and therefore CD = CQ, and PC + CQ = PC + CD = PD. Now if any other point, as E, be taken in AB, and EQ, ED, be joined, it can be proved, in the same manner, that the triangles BEQ, BED, are everyway equal; and hence EQ = ED. Hence PE + EQ = PE + ED; but (I. 20) PE + ED is greater than PD, which is equal to PC + CQ; consequently PE + EQ is greater than PC + CQ. It can similarly be proved that PC + CQ is less than the sum of the distances of any other point in AB from P and Q; hence PC + CQ is a minimum.

EXERCISE II.—THEOREM.

If an eccentric point be taken in the diameter of a circle, of all the chords passing through this point, that is the least which is perpendicular to the diameter.

Let R be an eccentric point in the diameter MN of a circle, then of all the chords passing through R, that which is perpendicular to MN, namely, PQ, is the least.

For draw through R any other chord ST, and from the centre O draw OV perpendicular to ST; then in the right-angled triangle ORV, the hypotenuse OR is greater than the side OV; consequently (III. 15) the chord PQ is less than ST. In the same way it can be proved that PQ is less than any other chord passing through R; hence PQ is a minimum.

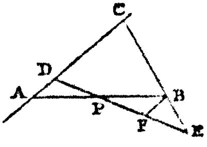


EXERCISE III.—THEOREM.

Of all triangles that have the same vertical angle, and whose bases pass through a given point, that whose base is bisected by the point is a minimum.

Let ACB be the given vertical angle of triangles whose bases pass through a given point P, and let AB, the base of the triangle ABC, be bisected in that point, then ABC is the least of all the triangles.

For let DCE be another triangle, and through B draw BF parallel to AC; then in the triangles APD, BPF, the vertical angles at P are equal, and the alternate angles A and PBF are also equal, and the side AP = PB; consequently the triangles are everyway equal, and therefore their areas are so; hence triangle PBE is greater than APD. To each of these unequal triangles add the quadrilateral CDPB, and the first sum, namely, the triangle CDE, is greater than the other sum, namely, the triangle ABC. The same can be proved of the triangle ABC, and any other whose vertical angle is C, and whose base passes through P; hence ABC is a minimum.

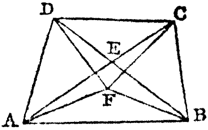


EXERCISE IV.—THEOREM.

The sum of the four lines drawn to the angular points of any quadrilateral from the intersection of the diagonals, is less than that of any other four lines similarly drawn from any other point.

Let the diagonals AC, BD, of the quadrilateral ABCD intersect in E, and let F be any other point; then AF + FB + FC + FD is greater than AE + EB + EC + ED.

For in the triangle AFC, AF + FC is greater than AC (I. 20); and similarly in triangle BFD, BF + FD is greater than BD; consequently AF + FC + BF + FD is greater than AC + BD; that is, than AE + EB + EC + ED. It can be similarly proved that the sum of the lines drawn from E to the angular points, is less than the sum of the distances of any other point from them; hence the sum of the distances of E from the angular points is a minimum.



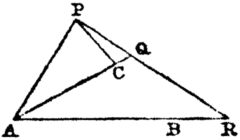
EXERCISE V.—PROBLEM.

To find a point in a given line, such that the difference of the lines drawn to it from two given points may be a maximum.

Let AB be a given line, and P, Q, two given points; it is required to find in AB a point such that the difference of its distances from P and Q may be a maximum.

Join PQ, and produce PQ to meet AB in R, and R is the required point.

For take any point A in AB; join AP, AQ, and in the line AQ cut off AC = AP, and join PC. Then (I. 5) in the triangle APC the angles at P and C are equal, and therefore each less than a right angle (I. 17); consequently angle PCQ is obtuse (I. 13, Cor.); and therefore the side PQ of the triangle PCQ must be greater than the side CQ. But PQ is the difference between PR and QR, and CQ is the difference between AP and AQ; hence the difference of the distances of R from P and Q exceeds the difference of the distances of



any point, as A, from P and Q; consequently PQ is a maximum, and R is the required point.

When the point A is such that AP = AQ, it is manifest that A is not the required point.

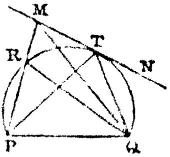
When AP is greater than AQ, interchange the letters P and Q, and the same proof will apply.

EXERCISE VI.—PROBLEM.

A straight line, and two points on the same side of it, being given, to find a point in it such that the angle contained by lines drawn from it to the given points shall be a maximum.

Let P, Q, be the given points, and MN the given line; to find a point T in that line such that angle PTQ shall be greater than any other angle PMQ formed at any other point in MN, by lines drawn to it from P and Q.

Join PQ, and describe (IV. Exercise 9th) through the points P, Q, a circle touching the line MN, and the point of contact T is the required point.



For draw PT, QT, and in MN take any other point M, and join MP and RQ. Then angle PRQ = PTQ (III. 22); but PRQ is greater than PMQ (I. 16); consequently PTQ is also greater than PMQ. It is similarly proved that PTQ exceeds any other angle formed at any point in MN by lines drawn from it to P and Q; therefore PTQ is a maximum.

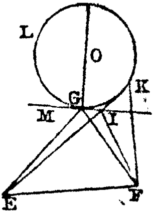
EXERCISE VII.—THEOREM.

The sum of two lines drawn from two given points to a point in the circumference of a given circle, is least when they are equally inclined to the radius or the tangent drawn to that point.

Let E, F, be the given points, and LGK the given circle; any two lines EG, GF, drawn to a point G in the circumference, such that they make equal angles with the radius

OG, are together less than the sum of EK, KF, drawn to any other point K of the circumference.

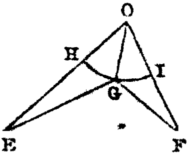
For join IF, and draw the tangent MGI. Then because EG, GF, make equal angles with OG, angle EGO = FGO; from these angles take away respectively the right angles OGM, OGI, and the remaining angles EGM, FGI, must be equal. Since, then, EG, GF, make equal angles with MI, they are together less than the lines EI, IF, drawn to I (1st Exercise); but KI and KF are greater than IF; to each add IE; therefore FK and KE are greater than IF and IE; but EG and GF are less than EI and IF; consequently EG and GF are still less than EK and KF. The same can be proved wherever K is taken in the circumference; hence EG + GF is a minimum.



EXERCISE VIII.—PROBLEM.

Given three points; to find a fourth such that the sum of its distances from the given points shall be a minimum.

Let E, O, F, be three points; to find a fourth G such that the sum of its distances EG, FG, and OG, from the three given points, shall be a minimum.



Let OG be the distance of the required point from O, whatever that may be, and describe the arc HGI, with radius OG and centre O; then the required point must lie in the arc HGI, and the sum of the other two lines EG, GF, must therefore be a minimum. But this is the case (last Exercise) when they make equal angles with the radius OG; that is, angle OGE = OGF. In the same manner, by describing an arc from E as a centre, and with a radius equal to the assumed distance of the required point G, it can be proved that OG and GF must make equal angles with EG. Consequently, when the three angles at G are equal, the sum of the three lines

EG, FG, and OG, is a minimum. Since the three angles are equal, each of them must be the third part of four right angles, or equal to a right angle with the third of a right angle; therefore, if on any two of the three lines EF, FO, OE, joining the points, segments of circles be described, each containing an angle equal to the third part of four right angles, that is, equal to double one of the angles of an equilateral triangle, they will intersect each other in the required point G.

PLANE LOCI.

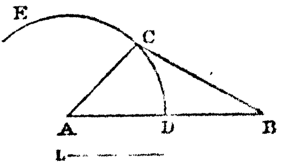
EXERCISE I.—PROBLEM.

To find the locus of the vertices of all the triangles that have the same base and one of their sides of a given length.

Let AB be the given base, and the line L the given length of one of the sides of a triangle; to find the locus of its vertex.

From A as a centre, with the radius AD = L, describe the arc DCE, and it is the required locus.

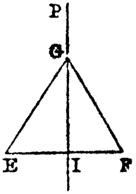
For take any point C in the arc, and join CB; then the triangle ABC has the given base AB, and one of its sides AC equal to the given line L; hence the point C is the vertex of a triangle fulfilling the assigned conditions of the problem. It can be similarly proved that any other point in the arc is also the vertex of a triangle satisfying the given conditions; consequently the arc DCE is the required locus.



II.—PROBLEM.

Find the locus of a point that is at equal distances from two given points.

Let E, F, be the two given points; it is required to find the locus of all the points that are equidistant from E and F.



Join EF, and bisect it perpendicularly by PI, and this line is the required locus.

For take any point G in IP, and join EG, GF. Then in the two triangles EIG, FIG, the side EI = IF, and IG is common, and the angles at I are right; hence (I. 4) the triangles are everyway equal, and therefore EG = GF. Now, G is any point in IP; hence IP is the required locus.

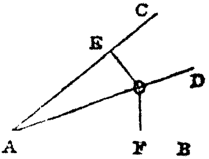
EXERCISE III.—PROBLEM.

To find the locus of a point that is equally distant from two given lines, either parallel, or inclined to one another.

Let AC, AB, be the given lines; it is required to find the locus of all the points that are equidistant from these lines.

Bisect the angle BAC by the line AD, and it is the required locus.

For from any point O in AD draw the perpendiculars OE, OF, and it can be proved (Book I. Ex. 8) that OE = OF; and therefore the point O is equidistant from the two given lines AB, AC; the same can be similarly proved of any other point in AD; therefore AD is the required locus.



When the given lines are parallel, draw a line perpendicular to one of them, and it will be perpendicular to the other (I, 29); then through the middle of this perpendicular draw a line parallel to the given lines, and it is the required locus.

For every point in the middle parallel is equidistant from each of the given parallels (I. 34, Cor. 1); and as one point in it, namely, the middle of the above perpendicular, is equidistant from the given parallels, therefore every point in it is equidistant from them both; it is therefore the required locus.

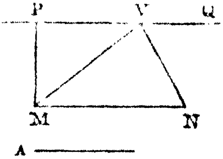
EXERCISE IV.—PROBLEM.

To find the locus of the vertices of all the triangles that have the same base and equal altitudes.

Let MN be the given base, and A the given altitude of a triangle; to find the locus of its vertex.

Draw MP perpendicular to MN, and make it = A, and through P draw PQ parallel to MN; and PQ is the required locus.

For in PQ take any point V, and join MV and NV; then (I. 34, Cor. 1) a perpendicular from V on MN would be = MP = A; and consequently, whatever point V is taken in PQ, the triangle thus formed has the given base and the given altitude; therefore PQ is the required locus of the vertices.



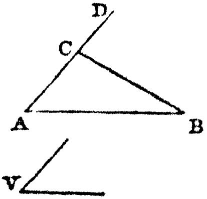
EXERCISE V.—PROBLEM.

To find the locus of the vertices of all the triangles that have the same base and one of the angles at the base the same.

Let AB be the given base of the triangles, and let one of the angles at the base be equal to the given angle V; to find the locus of the vertices.

Draw a line AD, making with AB an angle A = V, and AD is the required locus.

For take any point C in AD, and join BC; then the triangle ABC has the given base AB, and an angle A at the base equal to the given angle V; and the same can be proved wherever the point C is taken in AD; therefore AD is the required locus.



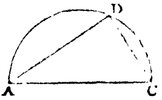
EXERCISE VI.—PROBLEM.

To find the locus of the angular point opposite to the hypotenuse of all the right-angled triangles that have the same hypotenuse.

Let AC be the given hypotenuse; to find the locus of the angular point of the right angle of all the right-angled triangles that can be constructed on AC.

On AC as a diameter describe the semicircle ADC, and it is the required locus.

For in the circumference take any point D, and join AD, DC; then (III. 31) angle D is a right angle; consequently the triangle ADC is right-angled, and it has the given hypotenuse AC; the same can be proved wherever the point D is taken in the circumference; hence the arc ADC is the required locus.



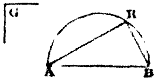
EXERCISE VII.—PROBLEM.

To find the locus of the vertices of all the triangles that have the same base and equal vertical angles.

Let AB be the given base, and G the given vertical angle of a triangle; to find the locus of its vertex.

On AB describe a circular segment AHB, containing an angle equal to given angle G (III. 33); and the arc AHB is the required locus.

For the triangle ABH has the given base AB, and its vertical angle H is equal to the given angle G; the same can be proved if H were any other point in the arc AHB; it is therefore the required locus.



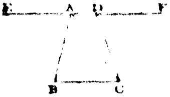
EXERCISE VIII.—PROBLEM.

To find the locus of the vertices of all the triangles that have the same base and equal areas.

Let BC be the given base, and the area of the triangle

ABC the given area; to find the locus of the vertices of triangles that have the same base and area.

Through A draw EF parallel to BC, and it is the required locus.



For in EF take any point D, and join BD, CD; then the area of triangle BCD is equal to that of ABC, and it has also the given base BC; since the same can be proved wherever the vertex D is taken in the line EF; therefore it is the required locus.

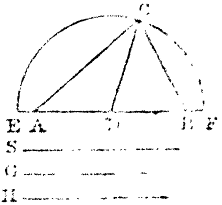
If the given area is that of any given rectilineal figure, it can be reduced to a parallelogram by I. 45, and the parallelogram thus found can be applied to half of the given base BC by I. 44; and then any triangle ABC being constructed on BC, having an altitude equal to that of the parallelogram, will be equal to the given area (I. 41).

EXERCISE IX.—PROBLEM.

To find the locus of the vertices of all triangles that have the same base, and the sum of the squares of their sides equal to a given square.

Let AB be the given base, and S the side of a square to which the sum of the squares of the two sides of each triangle must be equal; it is required to find the locus of the vertices of all the triangles that have AB for a base, and the sum of the squares of their two sides equal to the square on S.

Bisect AB in D, and then find a line G equal to the diagonal of a square of which AD is the side (I. 46); next find a line H whose square is equal to the difference between the squares of S and G; and then find a line CD equal to the side of a square of which H is the diagonal; which is done by describing on H, as a base, a triangle having each of the angles at the base half a right angle (see fig. to Prop. K of Book V., where



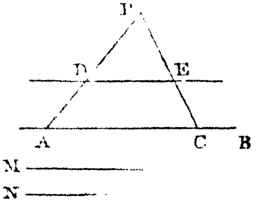
AC is the diagonal of a square, and AB, BC, the sides); and lastly, from D as a centre, with the radius CD, describe the semicircle ECF, and it is the required locus.

For $AC^2 + CB^2 = 2\,AD^2 + 2\,CD^2$ (II. A);
and $2\,AD^2 = G^2$ (I. 46 and 47);
also $2\,CD^2 = H^2$;
consequently $AC^2 + CB^2 = G^2 + H^2$.
But $H^2 = S^2 - G^2$, and adding G^2 to these equals,
$G^2 + H^2 = S^2$;
hence also $AC^2 + CB^2 = S^2$.
Therefore the sum of the squares of the sides of the triangle ABC is equal to the given square of S, and it has the given base; and since the same can be similarly proved wherever the point C is assumed in the arc ECF; therefore it is the required locus.

EXERCISE X.—PROBLEM.

If straight lines drawn from a given point to a given line be cut in a given ratio, to find the locus of the point of section.

Let P be a given point, AB a given line, and M, N, two lines in a given ratio; to find the locus of the point of section of lines drawn from P to AC, when they are divided in the given ratio.



Draw any line PA from P to AB, and cut it in D in the given ratio; that is, so that PD : DA = M : N; through D draw DE parallel to AB, and it is the required locus.

For in DE take any point E; join PE, and produce PE to C; then since DE is parallel to AC, therefore (VI. 2)

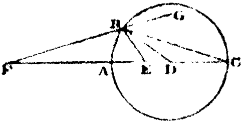
$$PD : DA = PE : EC;$$

but PD : DA = M : N by construction;
hence PE : EC = M : N by equal ratios (V. 11);
and as E is any point in DE, therefore DE is evidently the required locus.

EXERCISE XI.—PROBLEM.

To find the locus of the vertices of all the triangles that have the same base and the ratios of their sides equal.

Let FE be the base of the triangle, and let it be cut in A, so that FA : AE is the ratio of the sides (VI. 10).



Produce FD to C, so that FC : CE may = FA : AE (VI. Ex. 5); on AC as a diameter describe the circle ABC, and it is the required locus.

For, by construction, $FC : CE = FA : AE$;
hence, by alternation, $FC : FA = CE : AE$;
by mixing (V. G), $FC + FA : FC - FA = CE + AE : CE - AE$;
or $2\,DF : 2\,AD = 2\,AD : 2\,DE$;
wherefore (V. 15) $DF : AD = AD : DE$;
hence (VI. 16) $ED \cdot DF = AD^2$.

It can therefore be proved, exactly as in Prop. F of Book VI., that $FB : BE = FA : AE$;
that is, the ratio of the sides is the given ratio of FA : AE; and as B is any point in the circumference of the circle ABC, therefore it is the required locus.

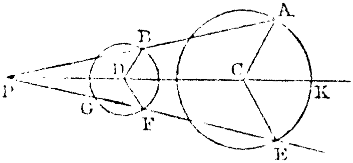
EXERCISE XII.—THEOREM.

If a straight line drawn from a given point, and terminating in the circumference of a given circle, be cut in a given ratio, the locus of the point of section is also the circumference of a given circle.

Let AKE be a given circle, P a given point, and PA any line drawn from P, and terminating in the circumference AKE of the given circle; let PA be cut in B, so that PA : PB in a given ratio; then the locus of the point B is also a circle.

For join A and the centre C of the given circle; draw

PCK through the centre C, and BD parallel to AC; then a circle described from D as a centre, with radius DB, is the required locus BGF.



For (VI. 4) $PA : PB = AC : BD$; wherefore $AC : BD$ is the given ratio; and if any other point E be taken in the circumference of the circle AKE, and the radius CE be drawn, and DF parallel to it, the line joining P and E, namely, PE, will pass through F.

For $\qquad PD : PC = DB : CA$;
therefore $\qquad PD : PC = DF : CE$.

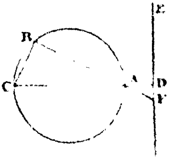
Now, if PE cut DF in any other point than F, namely, F′ (not shown in the figure); then, since in the triangles PDF′, PCE, the angles at D and C are equal (I. 29), and P is common, they are equiangular;
and consequently $PD : PC = DF' : CE$;
whence $\qquad DF : CE = DF' : CE$;
and therefore $DF = DF'$, or the point F′ coincides with F. Now E is any point in the circumference AKE; wherefore BGF is the required locus.

EXERCISE XIII.—THEOREM.

If one of the extremities of straight lines drawn through a given point be terminated in a given straight line, and their other extremities be determined, so that the rectangle under the segments of each line is equal to a given rectangle, the locus of these extremities will be the circumference of a circle; and if one of the extremities be terminated in the circumference of a circle, the locus of the other extremities is a straight line.

Let A be the given point, and EF the given line; then if any line, as BF, be terminated at one extremity F in the line EF, and if the rectangle $BA \cdot AF$ is to be equal to a given rectangle, namely, $DA \cdot AC$; then the locus of the point B will be a circle.

Let DC be perpendicular to EF, and on AC describe a circle ABC, and it is the required locus.



For since $FA \cdot AB = DA \cdot AC$, therefore (VI. 16) $FA : DA = AC : AB$; but angles DAF, BAC, are equal (I. 15); hence (VI. 5) the triangles ADF, ABC, are similar, and hence angle $ABC = ADF$; but ADF is a right angle, consequently ABC is also a right angle. But whatever be the direction of BF, the same can be proved; and as the angle in a semicircle is a right angle, the point B will always lie in the circumference of a semicircle of which AC is the base; consequently the circle ABC is the required locus.

Again, if the rectangle $FA \cdot AB$ is always $= DA \cdot AC$, and if the extremity B of the line BF is terminated in the circumference of a circle of which AC is the diameter, its other extremity F will be terminated in a straight line EF, perpendicular to CD.

For it can be proved, exactly as above, that the triangles ADF, ABC, are similar, and therefore angle $ADF = ABC$; but angle B is right (III. 31); consequently angle ADF is a right angle. Whatever be the direction of the line BF, it can be similarly proved that the side DF is perpendicular to AD; consequently the locus of F is the straight line EF.

Whatever rectangle is given, if M and N are its sides, then, assuming AD for one side of another equal rectangle, the side AC can then be found (VI. 12, and VI. 16).

PORISMS.

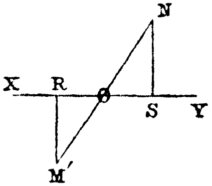
EXERCISE I.—PROBLEM.

Two points being given, to find a third, through which any straight line being drawn, the perpendiculars upon it from the two given points shall be equal.

Let M, N, be the given points, to find a third, as O, through which any line XY being drawn, the perpendiculars MR, NS, drawn upon it from M and N may be equal.

Join MN, and bisect MN in O, and O is the required point.

For draw any line XY through O, and the perpendiculars MR, NS, upon it. Then in the two triangles MOR, NOS, the angles at O are equal (I. 15), and the angles at R and S are also equal, being right, and the side MO = ON by construction; hence (I. 26) the triangles are everyway equal, and therefore MR = NS. The same can be proved of the perpendiculars from M and N on any other line through O; hence it is the required point.



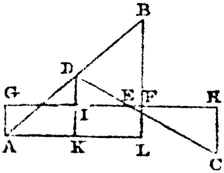
EXERCISE II.—PROBLEM.

Three points being given, to find a fourth, through which any straight line being drawn, the sum of the perpendiculars upon it from two of the given points on one side of it, shall be equal to the perpendicular on it from the third point.

Let A, B, C, be three points, to find a fourth E, through which if any line GH be drawn, the sum of the perpendiculars AG, CH, from the two points on one side of it, shall be equal to the perpendicular BF from the point on the other side of it.

Bisect AB in D, join CD, and cut CD in E, so that CE : ED = 2 : 1, and E is the required point.

For draw AL parallel to GH, and draw DI perpendicular to GH, and produce BF and DI to L and K. Then by similar triangles ABL, ADK, AB : AD = BL : DK; but AB = 2 AD, therefore BL = 2 DK; and taking FL = IK from both these equals, there-



fore $BF = 2DI + IK = 2DI + AG$, for $IK = AG$ (I. 34). Again, since CEH, DEI, are similar triangles,

$$EC : ED = CH : DI;$$

but, by construction, $EC = 2ED$; hence $CH = 2DI$. But $BF = 2DI + AG$; therefore $BF = CH + AG$; and therefore E is the required point.

COR.—Hence the perpendicular drawn from the middle of a given line on any line, is equal to half the sum or half the difference of the perpendiculars from its extremities on the same line, according as the extreme perpendiculars are on the same, or on different sides of the line.

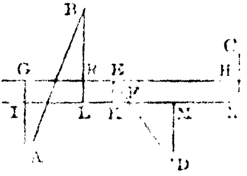
For AB being the given line, and GH any other line, it was shown that $BL = 2DK$;

therefore $BL - 2FL = 2DK - 2IK$;
or $BF - FL = 2DI$;
and since $FL = AG$, hence $BF - AG = 2DI$.

When the perpendiculars are all on one side of the line, the proof is more simple.

When there are four points given, to find their mean distance (see remark after the 2d Exercise in the Pl. Ge.)

Let A, B, C, D, be the four points; find E the point of mean distance of A, B, C, as above; join ED, and divide ED in F, so that $DF : FE = 3 : 1$; then F is the required point.



For draw any line IFN through F, and GH through E parallel to IN, and AG, BL, CN, DM, and EK, perpendiculars to these lines. Then, as above shown, $BR + CH = AG$;
hence, since IG, EK, RL, HN, are equal,

$$BR + RL + CH + HN = AG + 2EK;$$

or $BL + CN = AI + IG + 2EK = AI + 3EK$.
Now, from similar triangles DMF, EKF,

$$DF : FE = DM : EK;$$

but $DF = 3FE$ by construction; hence $DM = 3EK$.
Consequently $BL + CN = AI + DM$;

F is therefore the point of mean distance of A, B, C, and D.

The demonstration may be similarly extended in succession to five, six, or any number of points.

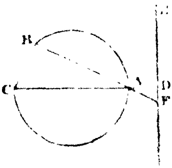
EXERCISE III.—PROBLEM.

A straight line and a circle being given, to find a point such that the rectangle under the segments of any straight line drawn through it, and limited by these, shall be equal to the rectangle contained by the external segment of a diameter perpendicular to the line, and produced to meet it, and the diameter itself.

Let ABC be the given circle, and EF the given line, and AC · AD the given rectangle, CD being perpendicular to EF.

The extremity of the diameter A is the required point.

For draw any line BAF through A, and limited by the circle and given line, and join BC. Then the triangles ADF, ABC, are similar, as the opposite angles at A are equal, and the right angle at D is equal to B, an angle in the semicircle ABC; hence the triangles are equiangular, and consequently AD : AF = AB : AC; and hence (VI. 16) AB · AF = AD · AC; and since the same can be proved of any other line through A, limited by the circle and given line, therefore A is the required point.

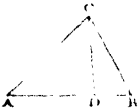


PLANE TRIGONOMETRY.

EXERCISE I.—THEOREM.

If in any triangle a perpendicular be drawn from the vertex upon the base, the segments of the base have the same ratio as the tangents of the parts into which the vertical angle is divided.

Let ABC be a triangle, and CD perpendicular to AB, then $AD : DB = \tan ACD : \tan BCD$.



For (Pl. Trig. Def. 4*) when CD is made radius, C being the centre, AD is the tangent of angle ACD, and DB is tangent of angle BCD; hence (Pl. Trig. 3)

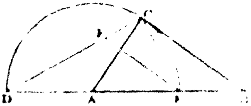
$$AD : DB = \tan ACD : \tan BCD.$$

EXERCISE II.—THEOREM.

The base of a triangle is to the sum of its two sides as the cosine of half the sum of the angles at the base to the cosine of half their difference.

Let ABC be any triangle, and BC its base; then $BC : BA + AC = \cos \frac{1}{2} (B + C) : \cos \frac{1}{2} (B \sim C)$.

For it is proved in Prop. VI. Pl. Trig., that angle $AEC = \frac{1}{2} (B + C)$, $ECB = \frac{1}{2} (C - B)$, and that $BD = BA + AC$.



Now (Pl. Trig. 5) in triangle BCD,

$$BC : BD = \sin D : \sin BCD.$$

But DEC or AEC is the complement of D, since DCE is a right angle (Pl. Trig. Def. 8); and angle ECB is the complement of BCD, as it is its difference from the right angle DCE; consequently $\sin D = \cos DEC$, and $\sin BCD = \cos ECB$;

therefore $BC : BA + AC = \cos DEC : \cos ECB$.

EXERCISE III.—THEOREM.

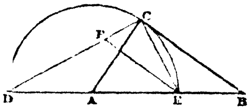
The base of a triangle is to the difference of its sides as the sine of half the sum of the angles at the base to sine of half their difference.

Let ABC be any triangle of which BC is the base, then $BC : AB - AC = \sin \frac{1}{2} (B + C) : \sin \frac{1}{2} (B \sim C)$.

---

* This is Definition 4th of Plane Trigonometry.

For it is proved, in the 6th Prop. of Pl. Trig., that BE = AB — AC, ECB = $\frac{1}{2}$(C—B), and AEC = $\frac{1}{2}$(B+C); also (Pl. Trig. Def. 2) BEC is the supplement of AEC; and therefore (Pl. Trig. Cor. 3 to Def.) sin AEC = sin BEC. Now in triangle BCE (Prop. 5) BC : BE = sin BEC : sin BCE;



or BC : BE = sin AEC : sin BCE;

. that is, BC : AB — AC = sin $\frac{1}{2}$ (B + C) : sin $\frac{1}{2}$ (C — B).

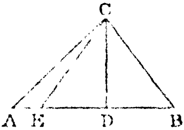
EXERCISE IV.—THEOREM.

The base of a triangle is to the difference of its segments as the sine of the vertical angle to the sine of the difference of the angles at the base.

Let ABC be a triangle, AE the difference of the segments of the base, and ACE the difference of the angles of the base, then

AB : AE = sin ACB : sin ACE.

For let CD be perpendicular to AB, make DE = DB, and join CE; then (I. 4) the triangles CDB, CDE, are everyway equal, and therefore angle CED = B. But CED = A + ACE; consequently angle ACE is the difference between CED and A, that is, between B and A; or ACE = B — A. Now, in triangle ABC (Trig. Prop. V.)



AB : AC = sin ACB : sin B;

and in triangle ACE, angle AEC is the supplement of CED; and hence (Pl. Trig. 3), sin AEC = sin CED or sin B; also

AE : AC = sin ACE : sin AEC or sin B;

whence, by direct equality (V. 22, or Ad. V. 11),

AB : AE = sin ACB : sin ACE;

but AE = AD—ED = AD—DB = the difference between the segments of the base, and ACE = B — A; therefore AB : AD — DB = sin ACB : sin (B — A).

EXERCISE V.—THEOREM.

Half the perimeter of a triangle is to its excess above the base, as the cotangent of half either of the angles at the base to the tangent of half the other angle.

Let ABC be a triangle, then, if $S = \frac{1}{2}(AB + BC + CA)$,

$$S : S - AB = \cot \tfrac{1}{2} A : \tan \tfrac{1}{2} B;$$

or

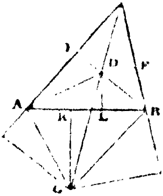
$$S : S - AB = \cot \tfrac{1}{2} B : \tan \tfrac{1}{2} A$$

For (VI. K) CF or $CE = S - AB = CM - AB$, $AK = BL$, $ED = DL$ (VI. H), and $MG = GK$.

Now, in triangle AGK, when AK is radius, GK is tan GAK; therefore

$$AK : GK = \text{radius} : \tan GAK.$$

So, in triangle BDL, when BL is radius, $BL : DL = \text{radius} : \tan DBL$. But $AK = BL$; wherefore, by direct equality (V. 22, or Ad. V. 11),



$$GK : DL = \tan GAK : \tan DBL.$$

Now (VI. H) DAL is the complement of GAK, and hence (Trig. Def. 8) $\tan GAK = \cot DAL$;

wherefore $GK : DL = \cot DAL : \tan DBL$.

But $GK = GM$, and $DL = DE$; also triangles CMG, CED, are similar;

wherefore $GK : DL$, or $GM : DE = CM : CE$;

hence $CM : CE = \cot DAL : \tan DBL$.

Again (Trig. Cor. 4 from Def.) $\cot DAL \cdot \tan DAL = R^2 = \cot DBL \cdot \tan DBL$; whence (VI. 16)

$$\cot DAL : \tan DBL = \cot DBL : \tan DAL;$$

and consequently also $CM : CE = \cot DBL : \tan DAL$.

Now, $CM = S$, $CE = S - AB$, $DAL = \frac{1}{2} A$, and $DBL = \frac{1}{2} B$;

wherefore $S : S - AB = \cot \frac{1}{2} A : \tan \frac{1}{2} B$;

and $S : S - AB = \cot \frac{1}{2} B : \tan \frac{1}{2} A$.

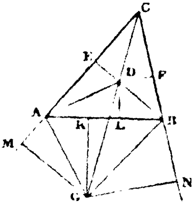
EXERCISE VI.—THEOREM.

The excess of half the perimeter of a triangle above the less side is to its excess above the greater, as the tangent of

half the greater angle at the base to the tangent of half the less.

Let ABC be a triangle, then $S - BC : S - AC = \tan \frac{1}{2} B : \tan \frac{1}{2} A$.

For $BK = BN = CN - BC = S - BC$ (VI. H); and $AK = AM = CM - AC = S - AC$; also in triangles AGK, BGK, when GK is radius, AK, KB, are tangents of the angles AGK, BGK, respectively, or cotangents of GAK and GBK, the complements of the former; hence $GK : KB =$ radius : cot GBK, and $GK : AK =$ radius : cot GAK; wherefore, by direct equality,



$$BK : AK = \cot GBK : \cot GAK.$$

But (VI. H) angle DAL is the complement of GAK, and DBL is that of GBK; hence (Trig. Def. 8)

$$BK : AK = \tan DBL : \tan DAL;$$

or $$S - BC : S - AC = \tan \tfrac{1}{2} B : \tan \tfrac{1}{2} A.$$

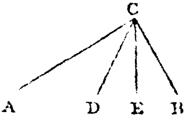
EXERCISE VII.—THEOREM.

In a right-angled triangle, radius is to the sine of double one of the acute angles as the square of half the hypotenuse to the area of the triangle.

Let ACB be a right-angled triangle, of which C is the right angle; then, if AB be bisected in D,

$$\text{radius} : \sin 2A = AD^2 : \text{area of } ABC.$$

For join CD, and draw CE perpendicular to AB; then it can be proved, exactly as in Ex. 22 of Book I., that $CD = AD$; hence (I. 5) angle $ACD = A$; but (I. 32) angle $CDE = A + ACD$; wherefore $CDE = 2A$. Now in triangle CDE, when CD is radius, CE is sin CDE; wherefore $CD : CE =$ radius : sin CDE or sin 2 A. Now if CD, CE, be the bases of two rectangles, whose altitudes are equal to AD, then (VI. 1)



$$CD : CE = AD \cdot CD : AD \cdot CE.$$

But $AD \cdot CD = AD^2$, and $AD \cdot CE$ = area of triangle ABC; wherefore, from equal ratios,

$$\text{radius} : \sin 2A = AD^2 : \text{area of } ABC.$$

EXERCISE VIII.—THEOREM.

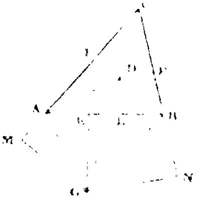
Radius is to the tangent of half the vertical angle of a triangle, as the rectangle under half the perimeter and its excess above the base, to the area of the triangle.

Let ABC be a triangle; then, assuming S = half the sum of its sides,

$$\text{radius} : \tan \tfrac{1}{2} C = S (S - AB) : ABC.$$

ABC denoting the area of the triangle.

For angle $DCE = \frac{1}{2} C$, and when CE is radius, ED is tangent of angle DCE (Trig. Def. 4); hence radius : tan DCE = CE : ED. Now if CE, ED, are the bases of two rectangles, of which CM is the altitude, then (VI. 1)



$$CE : ED = CM \cdot CE : ED \cdot CM.$$

But (VI. L) $ED \cdot CM$ = triangle ABC, $CM = S$ (VI. K), and $CE = S - AB$ (VI. K); wherefore

$$CE : ED = S (S - AB) : \text{triangle } ABC;$$

and hence $\quad \text{radius} : \tan \frac{1}{2} C = S (S - AB) : ABC.$

END OF KEY TO PLANE GEOMETRY.

Edinburgh:
Printed by W. and R. Chambers.